Global Asymptotic Stability of Recurrent Neural Networks With Multiple Time-Varying Delays

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***Abstract*—In this paper, several sufficient conditions are established for the global asymptotic stability of recurrent neural networks with multiple time-varying delays. The Lyapunov–Krasovskii stability theory for functional differential equations and the linear matrix inequality (LMI) approach are employed in our investigation. The results are shown to be generalizations of some previously published results and are less conservative than existing results. The present results are also applied to recurrent neural networks with constant time delays.**

***Index Terms*—Recurrent neural networks, global asymptotic stability, multiple time-varying delays, linear matrix inequality (LMI), Lyapunov–Krasovskii functional.**

I. Introduction

RECENTLY, dynamical recurrent neural networks have attracted considerable attentions as they are proved to be essential in applications such as classification of patterns, associative memories, and optimization. In particular, the recurrent neural network model introduced by Hopfield in [10] has been studied extensively and has been successfully applied to optimization problems. Moreover, time delay is commonly encountered in the implementation of neural networks due to the finite speed of information processing, and it is frequently a source of oscillation and instability in neural networks. Therefore, the stability of delayed neural networks has become a topic of great theoretical and practical importance. When a neural network is designed to function as an associative memory, it is required that there exist many stable equilibrium points, whereas in the case of solving optimization problems, it is necessary that the designed neural network has a unique equilibrium point that is globally asymptotically stable. Correspondingly, it is of great interest to establish conditions that ensure the global asymptotic stability of a unique equilibrium point of a neural network. Recently, many researchers have studied the equilibria and stability properties of neural networks and established various sufficient conditions for the uniqueness and global asymptotic stability of the equilibrium point for recurrent neural networks [1], [2], [4]–[9], [11]–[31], in which [6], [9], [11], [14], [31] are for the case of time-varying delays and others are for the case of constant delays.

In this paper, we will establish several new sufficient conditions for the global asymptotic stability of the equilibrium point for recurrent neural networks with multiple time-varying delays or constant delays. Compared to some previously published results, results of this paper are less conservative and more general. The main contributions of the paper include the following.

1) First, we present some linear matrix inequality (LMI)-based stability results for a class of neural networks with different multidelays and with different activation functions. The concerned model regards the models in [11], [14], [21], and [22] as special cases. Our present results can directly be applied to the models studied in [11], [14], [21], and [22]. In these works, stability results are obtained mainly in the form of M-matrix or algebraic inequality. No LMI-based results have been reported in the existing literature. On the other hand, because the LMI-based results consider the sign difference of the elements in connection matrices, neuron's excitatory and inhibitory effects on the neural network have been considered, which overcome the shortcomings of the results based on M-matrix and algebraic inequality. Correspondingly, results obtained using LMI techniques will be less conservative.
2) Many existing stability results in the form of LMI are special cases of the present results. Therefore, comparing to existing results, the present results are less conservative and have wider fields of applications.

This paper is organized as follows. In Section II, we introduce some notation, problem description, and preliminaries. In Section III, we establish main results of this paper and some corollaries. In Section IV, an example is included to demonstrate the applicability of the present results. In Section V, some pertinent remarks are provided to conclude this paper.

II. Problem Statement and Preliminaries

Throughout the paper, let $B^T, B^{-1}, \lambda_M(B), \lambda_m(B)$, and $\|B\| = \sqrt{\lambda_M(B^TB)}$ denote the transpose, the inverse, the largest eigenvalue, the smallest eigenvalue, and the Euclidean norm of a square matrix B, respectively. Let $B \geq 0 (B > 0, B < 0)$ denote a semipositive (positive, negative) definite symmetric matrix, respectively. Let I and 0 denote

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the identity matrix and the zero matrix with appropriate dimensions, respectively. Time-varying delays $\tau_{ij}(t)$ are assumed to be bounded, $0 \leq \tau_{ij}(t) \leq \upsilon_{ij}$, where υ_{ij} are constants. Let $\rho_i = \max_{1\leq j\leq n}(\upsilon_{ij})$, and $\rho = \max_{1\leq i\leq N}(\rho_i)$. $\dot{\tau}_{ij}(t)$ denotes the rate of change of $\tau_{ij}(t), i = 1,2,\ldots,N, j = 1,2,\ldots,n$.

Now, we consider the following neural network model with different multiple time-varying delays:

$$\frac{du_i(t)}{dt} = -a_i u_i(t) + \sum_{j=1}^{n} w_{ij}\bar{g}_j(u_j(t)) + \sum_{k=1}^{N}\sum_{j=1}^{n} w_{ij}^k \bar{f}_j(u_j(t-\tau_{kj}(t))) + U_i \quad (1)$$

or in a matrix-vector format

$$\frac{du(t)}{dt} = -Au(t) + W\bar{g}(u(t)) + \sum_{k=1}^{N} W_k \bar{f}(u(t-\bar{\tau}_k(t))) + U \quad (2)$$

where $u(t) = [u_1(t), u_2(t), \ldots, u_n(t)]^T$ is the neuron state vector, n is the number of neurons, $A = \operatorname{diag}(a_1, a_2, \ldots, a_n)$ with $a_i > 0, W = (w_{ij})_{n\times n} \in \Re^{n\times n}$ and $W_k = (w_{ij}^k)_{n\times n} \in \Re^{n\times n}$ are the connection weight matrix and delayed connection weight matrices, respectively, N denotes the number of delayed connection matrices, $\bar{\tau}_k(t) = [\tau_{k1}(t), \tau_{k2}(t), \ldots, \tau_{kn}(t)]^T, \tau_{kj}(t) \geq 0$, $\bar{g}(u(t)) = [\bar{g}_1(u_1(t)), \bar{g}_2(u_2(t)), \ldots, \bar{g}_n(u_n(t))]^T$, $\bar{f}(u(t-\bar{\tau}_k(t))) = [\bar{f}_1(u_1(t-\tau_{k1}(t))), \bar{f}_2(u_2(t-\tau_{k2}(t))), \ldots, \bar{f}_n(u_n(t-\tau_{kn}(t)))]^T$, $k = 1,2,\ldots,N$, $j = 1,\ldots,n$, and $U = [U_1, U_2, \ldots, U_n]^T$ denotes the external input vector.

Assumption 2.1: The activation functions $\bar{g}_j(u_j(t))$ and $\bar{f}_j(u_j(t))$ satisfy the following conditions, respectively:

$$0 \leq \frac{\bar{g}_j(\xi) - \bar{g}_j(\zeta)}{\xi - \zeta} \leq \delta_j, \quad 0 \leq \frac{\bar{f}_j(\xi) - \bar{f}_j(\zeta)}{\xi - \zeta} \leq l_j$$

for arbitrary $\xi, \zeta \in \Re$ and $\xi \neq \zeta$, and for some positive constant $\delta_j > 0, l_j > 0, j = 1,2,\ldots,n$. ■

Let $\Delta = \operatorname{diag}(\delta_1, \delta_2, \ldots, \delta_n), L = \operatorname{diag}(l_1, l_2, \ldots, l_n)$. Obviously, Δ and L are nonsingular.

In the proof of the main results of this paper, we require the following lemmas.

Lemma 2.1: Let X and Y be two real vectors with appropriate dimensions, and let Q and Π be two matrices with appropriate dimensions, where $Q > 0$. Then, for any two positive constants $m > 0$ and $l > 0$, the following inequality holds:

$$-mX^TQX + 2lX^T\Pi Y \leq l^2 Y^T \Pi^T (mQ)^{-1} \Pi Y.$$

This lemma can be proved as follows:

$$\begin{aligned} - mX^TQX &+ 2lX^T\Pi Y \\ &= -[(mQ)^{1/2}X - (mQ)^{-1/2}(l\Pi)Y]^T \\ &\quad \times[(mQ)^{1/2}X - (mQ)^{-1/2}(l\Pi)Y] + l^2Y^T\Pi^T(mQ)^{-1}\Pi Y \\ &\leq l^2Y^T\Pi^T(mQ)^{-1}\Pi Y. \end{aligned}$$

Lemma 2.2 (see [7] and [30]): If a map $H(u) \in C^0$ satisfies the following conditions:

1) $H(u)$ is injective on $\Re^n$;
2) $\|H(u)\| \to \infty$ as $\|u\| \to \infty$;

then $H(u)$ is a homeomorphism of $\Re^n$. ■

III. Global Asymptotic Stability Results

Prior to the global asymptotic stability analysis for recurrent neural networks (2), we first establish the following result, which guarantees the existence and uniqueness of the equilibrium point of neural networks (2).

Proposition 3.1 (See Appendix I for a Proof): Suppose that $0 \leq \dot{\tau}_{ij}(t) \leq \mu_{ij} < 1$. If there exist positive–definite symmetric matrix $P > 0$, positive diagonal matrices $H = \operatorname{diag}(h_1, \ldots, h_n), M = \operatorname{diag}(m_1, \ldots, m_n), H_g, M_f$, and $Q_i = \operatorname{diag}(q_{i1}, \cdots, q_{in}), i = 1,2,\cdots,N$, such that the LMI in (3), shown at the bottom of the page, holds, then the neural network (2) has a unique equilibrium point for a given U, where $*$ is the symmetric part of the corresponding element in a matrix, and $\gamma_i = \min_{1\leq j\leq n}(1-\mu_{ij}), i = 1,2,\ldots,N$. ■

In the following, we assume that neural network (2) has an equilibrium point, namely, $u^* = [u_1^*, u_2^*, \ldots, u_n^*]^T$, for a given U. By linear transformation $x(t) = u(t) - u^*$, neural network (2) is converted into the following form:

$$\begin{aligned} \frac{dx(t)}{dt} &= -Ax(t) + Wg(x(t)) + \sum_{k=1}^{N} W_k f(x(t-\bar{\tau}_k(t))) \\ x(t) &= \phi(t), t \in [-\rho, 0] \end{aligned} \quad (4)$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$ is the state vector of the transformed model

$$\begin{aligned} g(x(t)) &= [g_1(x_1(t)), g_2(x_2(t)), \ldots, g_n(x_n(t))]^T \\ g_j(x_j(t)) &= \bar{g}_j(x_j(t)+u_j^*) - \bar{g}_j(u_j^*) \quad \text{with } g_j(0) = 0 \\ f(x(t-\bar{\tau}_k(t))) &= [f_1(x_1(t-\tau_{k1}(t))), f_2(x_2(t-\tau_{k2}(t))), \\ &\qquad \ldots, f_n(x_n(t-\tau_{kn}(t)))]^T \\ f_j(x_j(t-\tau_{kj}(t))) &= \bar{f}_j(x_j(t-\tau_{kj}(t)) + u_j^*) - \bar{f}_j(u_j^*) \end{aligned}$$

$$\Xi_w = \begin{bmatrix} -PA - AP & PW + H_g\Delta - AH & M_fL - AM & PW_1 & PW_2 & \cdots & PW_N \\ * & HW + W^TH - 2H_g & W^TM & HW_1 & HW_2 & \cdots & HW_N \\ * & * & \sum_{i=1}^{N} Q_i - 2M_f & MW_1 & MW_2 & \cdots & MW_N \\ * & * & * & -\gamma_1 Q_1 & 0 & \cdots & 0 \\ * & * & * & * & -\gamma_2 Q_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & * & * & \cdots & -\gamma_N Q_N \end{bmatrix} < 0 \quad (3)$$

with $f_j(0) = 0$ and $\phi(t)$ is a continuous and bounded vector valued function with the supremum norm $\|\phi\| = \sup_{-\rho\le\theta\le 0}\|\phi(\theta)\|$. By Assumption 2.1, we can see that $0 \le (g_j(x_j(t))/x_j(t)) \le \delta_j$ and $0 \le (f_j(x_j(t))/x_j(t)) \le l_j$ for $\forall x_j(t) \ne 0, j = 1,2,\ldots,n, k = 1,2,\ldots,N$.

Clearly, the equilibrium point u^* of neural network (2) is globally asymptotically stable if the zero solution of model (4) is globally asymptotically stable.

A. Recurrent Neural Networks With Different Multiple Time-Varying Delays

Theorem 3.1 (See Appendix II for a Proof): Suppose that $0 \le \dot{\tau}_{ij}(t) \le \mu_{ij} < 1$. If the condition (3) in Proposition 3.1 holds, then the equilibrium point of neural network (2) is globally asymptotically stable, independent of the magnitude of time delays. ∎

For the following model:

$$\frac{du_i(t)}{dt} = -a_i u_i(t) + \sum_{j=1}^{n} w_{ij}\bar{g}_j(u_j(t)) + \sum_{j=1}^{n} w_{ij}^1 \bar{f}_j(u_j(t-\tau_j(t))) + U_i \quad (5)$$

which is a special case of model (1) [if let we $N = 1$ in (1)], we can easily obtain the following result.

Corollary 3.1: Suppose that $0 \le \dot{\tau}_j(t) \le \mu_j < 1$. If there exist a positive–definite symmetric matrix $P > 0$, positive diagonal matrices $H = \mathrm{diag}(h_1,\ldots,h_n)$, $M = \mathrm{diag}(m_1,\ldots,m_n)$, H_g, M_f, and $Q = \mathrm{diag}(q_1,\ldots,q_n)$, such that the following LMI holds:

$$\begin{bmatrix} -PA-AP & \Theta_{12} & M_f L - AM & PW_1 \\ * & \Theta_{22} & W^T M & HW_1 \\ * & * & Q-2M_f & MW_1 \\ * & * & * & -\tilde{\gamma}Q \end{bmatrix} < 0 \quad (6)$$

then the equilibrium point of neural network (5) is globally asymptotically stable, independent of the magnitude of time delays, where * represents the symmetric parts of the corresponding element in a matrix, $\Theta_{12} = PW + H_g\Delta - HA$, $\Theta_{22} = HW + W^T H - 2H_g$, and $\tilde{\gamma} = \min_{1\le j\le n}(1-\mu_j)$.

Proof: In a similar manner to the proof of Theorem 3.1, we can easily obtain the result. ∎

Remark 3.1: The global asymptotic stability problem for model (5) was studied in [6]. The existence and uniqueness of the equilibrium point, which is stated in [6, Th. 1 and 2], are proved by using homotopic map and M-matrix theory. The global asymptotic stability results, which are stated in [6, Th. 3–5], are derived using the M-matrix theory and some algebraic inequality techniques. We emphasize that the techniques used in [6] ignored the neuron's excitatory and inhibitory effects on neural networks. In contrast, we use homeomorphism map and LMI technique to prove the existence and uniqueness of the equilibrium point, the global asymptotic stability conditions are established in the form of LMIs, and our derivation considers the neuron's excitatory and inhibitory effects on neural networks. Obviously, the results in [6] and this paper are different, and cannot be replaced by each other. ∎

Example 3.1: Consider the neural network (5) with constant delays, where $A = \mathrm{diag}(1,1)$, $W = (w_{ij})_{2\times 2} = \begin{bmatrix} -c & c \\ -c & -c \end{bmatrix}$, $W_1 = (w_{ij}^1)_{2\times 2} = \begin{bmatrix} -c & -c \\ -c & c \end{bmatrix}$, and $c \ge 0$, $\Delta = L = 0.5\mathrm{diag}(1,1)$. In this example, $c > 0$ in connection matrices W and W_1 represents the excitatory connection of neurons, and $-c < 0$ in connection matrices W and W_1 represents the inhibitory connection of neurons.

Theorem 1 or 3 in [6] requires that

$$\begin{aligned} &|A| - |W|\Delta - |W_1|L \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 0.5\begin{bmatrix} |-c| & |c| \\ |-c| & |-c| \end{bmatrix} - 0.5\begin{bmatrix} |-c| & |-c| \\ |-c| & |c| \end{bmatrix} \\ &= \begin{bmatrix} 1-c & -c \\ -c & 1-c \end{bmatrix} > 0. \end{aligned}$$

Clearly, Theorem 1 or 3 in [6] holds only for $0 \le c < 0.5$, while our Corollary 3.1 holds for $0 \le c \le 1.25$. Obviously, for $0.5 \le c \le 1.25$, Theorem 1 or 3 in [6] is not satisfied. The main reason for Theorems 1 and 3 of [6] to fail is the neglect of the sign differences of elements in connection matrices W and W_1 because of the use of the absolute values in the previous expression. In general, the inhibitory effect of a neuron can stabilize a neural network, while the excitatory effect may destabilize a neural network. ∎

When $\bar{g}_j(\cdot) = \bar{f}_j(\cdot)$ in model (1), then it becomes the following model:

$$\frac{du_i(t)}{dt} = -a_i u_i(t) + \sum_{j=1}^{n} w_{ij}\bar{g}_j(u_j(t)) + \sum_{k=1}^{N}\sum_{j=1}^{n} w_{ij}^k \bar{g}_j(u_j(t-\tau_{kj}(t))) + U_i \quad (7)$$

or in a matrix-vector format

$$\frac{du(t)}{dt} = -Au(t) + W\bar{g}(u(t)) + \sum_{k=1}^{N} W_k \bar{g}(u(t-\bar{\tau}_k(t))) + U. \quad (8)$$

Different from Proposition 3.1, we give another sufficient condition to ensure the existence and uniqueness of the equilibrium point of model (8).

Proposition 3.2 (See Appendix III for a Proof): Suppose that $0 \le \dot{\tau}_{ij}(t) \le \mu_{ij} < 1$. If there exist positive diagonal matrices $P = \mathrm{diag}(p_1,\ldots,p_n)$ and $Q_i = \mathrm{diag}(q_{i1},\ldots,q_{in})$, $i = 1,2,\ldots,N$, such that the following inequality holds:

$$-2PA\Delta^{-1} + PW + W^T P + \sum_{i=1}^{N}\frac{1}{\gamma_i}PW_iQ_i^{-1}W_i^T P + \sum_{i=1}^{N} Q_i < 0 \quad (9)$$

where $\gamma_i = \min_{1\le j\le n}(1-\mu_{ij})$, $i=1,2,\dots,N$, then the neural network (8) has a unique equilibrium point for a given U. ∎

By Proposition 3.2, neural network (8) has an equilibrium point, namely, $u^* = [u_1^*, u_2^*, \dots, u_n^*]^T$, for a given U. By linear transformation $x(t) = u(t) - u^*$, neural network (8) is converted into the following form:

$$\frac{dx(t)}{dt} = -Ax(t) + Wf(x(t)) + \sum_{k=1}^{N} W_k f(x(t-\bar{\tau}_k(t)))$$
$$x(t) = \phi(t), t \in [-\rho, 0] \tag{10}$$

where the notation is the same as that in (4).

Clearly, the equilibrium point u^* of neural network (8) is globally asymptotically stable if the zero solution of model (10) is globally asymptotically stable. Next, we will establish a sufficient condition to ensure the global asymptotic stability of model (10), which is different from Theorem 3.1.

Theorem 3.2 (See Appendix IV for a Proof): Suppose that $0 \le \dot{\tau}_{ij}(t) \le \mu_{ij} < 1$. If the condition (9) in Proposition 3.2 holds, then the equilibrium point of neural network (8) is globally asymptotically stable, independent of the magnitude of time delays.

In the following, Theorem 3.2 will be modified for the case of constant delays.

Corollary 3.2: Suppose that $\tau_{ij}(t) = \tau_{ij}$, i.e., $\dot{\tau}_{ij}(t) = 0$. If there exist positive diagonal matrices $P = \mathrm{diag}(p_1, \dots, p_n)$ and $Q_i = \mathrm{diag}(q_{i1}, \dots, q_{in}), i = 1,2,\dots,N$, such that the following inequality holds:

$$-2PA\Delta^{-1} + PW + W^TP + \sum_{i=1}^{N} PW_iQ_i^{-1}W_i^TP + \sum_{i=1}^{N} Q_i < 0 \tag{11}$$

then the equilibrium point of neural network (8) is globally asymptotically stable, independent of time delays.

Remark 3.2: Schur's Complement [3] can be stated as follows. The LMI

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} < 0$$

where $Q(x) = Q^T(x), R(x) = R^T(x)$ and $S(x)$ depends affinely on x, is equivalent to

$$R(x) < 0, \quad Q(x) - S(x)R^{\dagger}(x)S^T(x) < 0$$

where $R^{\dagger}(x)$ denotes the Moore–Penrose inverse of $R(x)$. By Schur's complement, the conditions in Theorem 3.2 and Corollary 3.2 can easily be expressed in the form of LMIs. For example, condition (9) in Theorem 3.2 can be converted into the following form of LMI:

$$\begin{bmatrix} \Theta & PW_1 & PW_2 & \cdots & PW_N \\ W_1^TP & -\gamma_1 Q_1 & 0 & \cdots & 0 \\ W_2^TP & 0 & -\gamma_2 Q_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ W_N^TP & 0 & 0 & \cdots & -\gamma_N Q_N \end{bmatrix} < 0$$

and condition (11) in Corollary 3.2 can be converted into the following form of LMI:

$$\begin{bmatrix} \Theta & PW_1 & PW_2 & \cdots & PW_N \\ W_1^TP & -Q_1 & 0 & \cdots & 0 \\ W_2^TP & 0 & -Q_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ W_N^TP & 0 & 0 & \cdots & -Q_N \end{bmatrix} < 0$$

where $\Theta = -2PA\Delta^{-1} + PW + W^TP + \sum_{i=1}^{N} Q_i$. Therefore, the results obtained in this paper can easily be checked. ∎

So far, many delayed neural network models have been studied in the literature. In [1], [25], and [30], the following neural network with single time delay $\tau \ge 0$ was studied:

$$\dot{u}_i(t) = -a_iu_i(t) + \sum_{j=1}^{n} w_{ij}^1 g_j(u_j(t-\tau)) + U_i. \tag{12}$$

A new delayed neural network model, called delayed cellular neural network model, has been introduced in [5], [17], [18], [20], [26], and [27], which is given by the following state equation:

$$\dot{u}(t) = -u(t) + Wg(u(t)) + W_1 g(u(t-\tau)) + U. \tag{13}$$

The version of (13) with multiple time delays, which was studied in [16], [19], [28], and [29], is in the following form:

$$\dot{u}(t) = -Au(t) + Wg(u(t)) + \sum_{k=1}^{N} W_k g(u(t-\tau_k)) + U \tag{14}$$

where $\tau_k \ge 0$ are scalars, $k = 1, \dots, N$.

On the other hand, in [8], different time delays $\tau_{ij} \ge 0$ were introduced into system (12) to obtain the following neural network model [8], [11]:

$$\dot{u}_i(t) = -a_iu_i(t) + \sum_{j=1}^{n} w_{ij}^1 g_j(u_j(t-\tau_{ij})) + U_i. \tag{15}$$

Another kind of neural systems of retarded functional differential equations considered in [14], [21], and [22], which generalizes both Hopfield neural network model and cellular neural network model, is in the following form:

$$\dot{u}_i(t) = -a_iu_i(t) + \sum_{j=1}^{n} w_{ij} g_j(u_j(t)) + \sum_{j=1}^{n} w_{ij}^1 g_j(u_j(t-\tau_{ij})) + U_i. \tag{16}$$

The neural network models (12)–(16) have attracted attention due to their promising potential for the task of classification, associative memories, parallel computation, and optimization. Such engineering applications rely crucially on the analysis of dynamical behavior of the designed neural networks. Therefore, the analysis of the network equilibrium point and the stability properties of the equilibrium points are of prime importance. Generally, the neural network models (12)–(16) can be divided into two different groups, i.e., model (14) and model (16). That is to say, models (12) and (13) are special cases of model (14), and model (15) is a special case of model (16). To the best of our knowledge, the analysis of models (14) and (16) has been dealt with independently, especially in the case of using the LMI approach, and it does not appear to have an integrated framework which covers both model (14) and model (16) in the literature.

Next, we will show that the model studied in this paper, i.e., model (7) or (8), can unify model (14) and model (16). Letting $\tau_{kj}(t) = \tau_k$ in (7), we obtain model (14). Next, we will show that model (16) can be changed to the form of (8). If we let B_k denote a square matrix, whose kth row is composed of the kth row of square matrix $W_1 = [w_{ij}^1]_{n\times n}$ and other rows are all zeros, and let $g(u(t-\bar{\tau}_k)) = [g_1(u_1(t-\tau_{k1})),\ldots,g_n(u_n(t-\tau_{kn}))]^T$, then model (16) can be written in a matrix–vector form as

$$\frac{du(t)}{dt} = -Au(t) + Wg(u(t)) + \sum_{k=1}^{n} B_k g(u(t-\bar{\tau}_k)) + U \tag{17}$$

where $\bar{\tau}_k = (\tau_{k1},\ldots,\tau_{kn})^T$, $k = 1,\ldots,n$. Obviously, model (17) is a special case of model (8). Therefore, the obtained results for model (7) or (8) in this paper can also be applied to models (12)–(16).

Remark 3.3: For the neural network model (16), to the best of our knowledge, no stability criterion in the form of LMI has been reported in the literature. We establish in this paper a stability criterion for model (7) or (8) in the form of LMI. Because model (16) is a special case of the neural network model (7) or (8) studied in our paper, the present results for model (7) or (8) represent a significant step for the stability analysis of the recurrent neural networks with delays and they can also be applied to model (16). Besides, results in this paper cannot be obtained using existing results in the literature. In the framework of Theorem 3.2, we will present some stability results for models (12)–(14) [which are special cases of model (19) to be introduced next] and give some comparisons between our results and some existing results. These comparisons will show that many existing stability results in the form of LMI are special cases of the results established in this paper. ■

Remark 3.4: Note that for models (12)–(14), the diagonal matrices Q_i in Corollary 3.2 can be relaxed to positive–definite symmetric matrices Q_i, $i = 1,\ldots,N$. Without loss of generality, for instance, let us consider model (14). By linear transformation $x(t) = u(t) - u^*$, model (14) is converted into the following form:

$$\dot{x}(t) = -Ax(t) + Wf(x(t)) + \sum_{k=1}^{N} W_k f(x(t-\tau_k))$$

where $f(x(t-\tau_k)) = [f_1(x_1(t-\tau_k)),\ldots,f_n(x_n(t-\tau_k))]^T$, and other notation is the same as that in (10). Now, we consider the following Lyapunov–Krasovskii functional:

$$V(x(t)) = (N+1)x^T(t)x(t) + 2\alpha\sum_{i=1}^{n} p_i \int_0^{x_i(t)} f_i(s)ds + \sum_{i=1}^{N}(\alpha+\beta_i)\int_{t-\tau_i}^{t} f^T(x(s))Q_i f(x(s))ds$$

where $Q_i > 0, i = 1,\ldots,N$, and other notation is the same as that in the proof of Theorem 3.2 in Appendix IV [see (75)]. In a similar manner to the proof of Theorem 3.2, we can obtain condition (9) except that $Q_i, i = 1,\ldots,N$, are positive–definite symmetric matrices. ■

B. Recurrent Neural Networks With Multiple Time-Varying Delays

In the following, we will consider the time-varying version of model (14) with time-varying delays, i.e.,

$$\dot{u}(t) = -Au(t) + Wg(u(t)) + \sum_{k=1}^{N} W_k g(u(t-\tau_k(t))) + U \tag{18}$$

where $\tau_k(t) \geq 0$, and other notation is defined the same way as that in (8). By linear transformation $x(t) = u(t) - u^*$, model (18) is changed to the following form:

$$\dot{x}(t) = -Ax(t) + Wf(x(t)) + \sum_{k=1}^{N} W_k f(x(t-\tau_k(t))). \tag{19}$$

Now, we establish another sufficient condition to guarantee the global asymptotic stability of the origin of model (19).

Theorem 3.3 (See Appendix V for a Proof): Suppose that $0 \leq \dot{\tau}_i(t) \leq \eta_i < 1$. Then, the origin of system (19) is globally asymptotically stable if there exist positive–definite symmetric matrices $P_i > 0, i = 1,\ldots,N$, and a positive diagonal matrix $D = \mathrm{diag}(d_1,\ldots,d_n) > 0$ such that the following condition holds:

$$\Omega = -2DA\Delta^{-1} + DW + W^TD + \sum_{i=1}^{N}\left(\frac{1}{1-\eta_i}DW_iP_i^{-1}W_i^TD + P_i\right) < 0. \tag{20}$$

Remark 3.5: When $A = I$, $N = 1$, $\tau_1(t) = \tau$, and $g_i(u_i(t)) = 0.5(|u_i(t)+1| - |u_i(t)-1|), i = 1,\ldots,n$, i.e., $\Delta = I$, model (18) has been studied in [2]. Theorem 3 in [2] requires the following: 1) $-(W + W^T + \beta I)$ is positive definite and 2) $\|W_1\| \leq \sqrt{2\beta}$ if $\beta \geq 1$, or $\|W_1\| \leq \sqrt{1+\beta}$ if $0 < \beta \leq 1$. In fact, from 2) we have 2') $-2I + W_1W_1^T/\beta \leq 0$ if $\beta \geq 1$ or $-I + W_1W_1^T/(1+\beta) \leq 0$ if $0 < \beta \leq 1$. Adding 1) and 2') together, we get

$$-2I + W + W^T + \beta I + W_1W_1^T/\beta < 0, \qquad \text{if } \beta \geq 1 \tag{21}$$

or

$$-I + W + W^T + \beta I + W_1W_1^T/(1+\beta) < 0, \qquad \text{if } 0 < \beta \leq 1. \tag{22}$$

In this case, (20) becomes

$$-2D + DW + W^TD + P_1 + DW_1P_1^{-1}W_1^TD < 0. \tag{23}$$

If we let $D = I$, $P_1 = \beta I$, or $P_1 = (1+\beta)I$, (23) can be transformed to (21) and (22), respectively, which shows that [2, Th. 3] is a special case of the present Theorem 3.3. Similarly, Theorem 1 in [2] and the main result in [20] are also special cases of Theorem 3.3 of this paper. ■

Remark 3.6: When $N = 1$, the delay-dependent stability criteria for (18) are established in [18]. Now, as far as the global asymptotic stability problem is concerned, we will summarize the relationships between the LMI-based delay-dependent stability criteria in [18] and Theorem 3.3 in this paper. For brevity of comparison, we consider the case of constant time delay for model (18) with $N = 1$. In this case, Theorem 3.3 is expressed as follows:

$$\Omega_a = -2DA\Delta^{-1} + DW + W^T D + DW_1 P_1^{-1} W_1^T D + P_1 < 0. \tag{24}$$

By Lemma 2.1 and for any vector $h(x) \neq 0$ with appropriate dimension, from (24), we have

$$\begin{aligned} h^T(x)\Omega_a h(x) &= h^T(x)\big[-2DA\Delta^{-1} + DW + W^T D \\ &\qquad + DW_1 P_1^{-1} W_1^T D + P_1\big] h(x) \\ &\leq h^T(x)\big[-2DA\Delta^{-1} + DWQ^{-1}W^T D + Q \\ &\qquad + DW_1 P_1^{-1} W_1^T D + P_1\big] h(x) \\ &= (\Delta^{-1}h(x))^T\big[-2\Delta DA + \Delta DWQ^{-1}W^T D\Delta \\ &\qquad + \Delta DW_1 P_1^{-1} W_1^T D\Delta \\ &\qquad + \Delta(P_1 + Q)\Delta\big]\Delta^{-1}h(x) \\ &\leq (\Delta^{-1}h(x))^T\big[-2\Delta DA + 2k\Delta D + \Delta DWQ^{-1}W^T D\Delta \\ &\qquad + e^{2k\rho}\Delta DW_1 P_1^{-1} W_1^T D\Delta \\ &\qquad + \Delta(P_1 + Q)\Delta\big]\Delta^{-1}h(x) \end{aligned} \tag{25}$$

for any $Q > 0$ and $k \geq 0$. Letting $D_a = \Delta D$, from (25), we have

$$h^T(x)\Omega_a h(x) \leq (\Delta^{-1}h(x))^T \Omega_b \Delta^{-1} h(x) \tag{26}$$

where

$$\begin{aligned} \Omega_b = &-D_a A - AD_a^T + 2kD_a + D_a WQ^{-1}W^T D_a^{-T} \\ &+ e^{2k\rho} D_a W_1 P_1^{-1} W_1^T D_a^T + \Delta(P_1 + Q)\Delta. \end{aligned} \tag{27}$$

$\Omega_b < 0$ is in the same form as the stability condition in [18, Th. 1]. (Note that D_a in [18, Th. 1] is positive definite.) Obviously, in the case of D_a being diagonal matrix, $\Omega_b < 0$ is a sufficient condition to ensure $\Omega_a < 0$, where we have used the fact that $\Omega_b < 0$ is equivalent to $\Delta\Omega_b\Delta < 0$ for nonsingular matrix Δ. That is to say, Theorem 1 in [18] is a special case of Theorem 3.3 of this paper in the case of D_a being a diagonal matrix. Similarly, Theorems 2 and 3 in [18] are also special cases of Theorem 3.3 of this paper in the case of D_a being a diagonal matrix. ■

Corollary 3.3: The origin of system (19) is globally asymptotically stable if there exist positive–definite symmetric matrices $\bar{P}_i > 0$ and positive diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$ such that the following condition holds:

$$\Omega_0 = -2DA\Delta^{-1} + DW + W^T D + \sum_{i=1}^{N}\left[\frac{1}{1-\eta_i} D\bar{P}_i^{-1} D + W_i^T \bar{P}_i W_i\right] < 0$$

where $0 \leq \dot{\tau}_i(t) \leq \eta_i < 1, i = 1, \ldots, N$.

Proof: Letting $P_i = W_i^T \bar{P}_i W_i$ in Theorem 3.3, and in a similar manner to the proof of Theorem 3.3, we can easily obtain the corollary. The details are omitted. ■

Remark 3.7: When $N = 1$, the global asymptotic stability for model (18) is studied in [17]. Theorems 2 and 4 in [17] are special cases of Corollary 3.3. For brevity of comparison, we consider the case of constant time delay for model (18) with $N = 1$. Then, Theorem 2 in [17] is restated as follows:

$$\Omega_1 = -2DA\Delta^{-1} + DW + W^T D + D\bar{P}_1^{-1} D + W_1^T \bar{P} W_1 < 0. \tag{28}$$

By Lemma 2.1, and for any vector $h(x) \neq 0$ with appropriate dimension, from (28), we have

$$\begin{aligned} h^T(x)\Omega_1 h(x) &= h^T(x)\big[-2DA\Delta^{-1} + DW + W^T D \\ &\qquad + D\bar{P}_1^{-1} D + W_1^T \bar{P} W_1\big] h(x) \\ &\leq h^T(x)\big[-2DA\Delta^{-1} + DQ_1^{-1} D + W^T Q_1 W \\ &\qquad + D\bar{P}_1^{-1} D + W_1^T \bar{P} W_1\big] h(x) \\ &= (\Delta^{-1}h(x))^T\big[-2\Delta DA + \Delta DQ_1^{-1} D\Delta + \Delta D\bar{P}_1^{-1} D\Delta \\ &\qquad + \Delta(W^T Q_1 W + W_1^T \bar{P} W_1)\Delta\big]\Delta^{-1}h(x) \end{aligned} \tag{29}$$

where $Q_1 > 0$. If we let $\bar{D} = \Delta D$ and $\bar{P}_1 = \beta\bar{D}$, where $\beta > 0$ is a scalar, from (29), we have

$$h^T(x)\Omega_1 h(x) \leq (\Delta^{-1}h(x))^T \Omega_2 \Delta^{-1} h(x)$$

where

$$\begin{aligned} \Omega_2 = &-\bar{D}A - A\bar{D}^T + \bar{D}Q_1^{-1}\bar{D} + \beta^{-1}\bar{D} \\ &+ \Delta(W^T Q_1 W + \beta W_1^T \bar{D} W_1)\Delta. \end{aligned} \tag{30}$$

$\Omega_2 < 0$ is in the same form as the condition in [17, Th. 1]. (Note that the matrix $\bar{D}$ in [17, Th. 1] is positive definite.) Obviously, $\Omega_2 < 0$ is only a sufficient condition to ensure $\Omega_1 < 0$, where we have used the fact that $\Omega_2 < 0$ is equivalent to $\Delta\Omega_2\Delta < 0$ for nonsingular matrix Δ. In the case of $\bar{D}$ being a diagonal matrix, Theorem 1 in [17] is a special case of Theorem 2 in [17]. Similarly, Theorem 3 in [17] is a special case of Theorem 4 in [17]. Therefore, all the results in [17] are special cases of the present Corollary 3.3 in the case of $\bar{D}$ being a diagonal matrix. ■

Remark 3.8: Global asymptotic stability problem for neural network model (18) with multidelays is studied in [19]. In a similar manner to the analysis in Remark 3.7, we can easily show that all results in [19] are special cases of Theorem 3.3 of this paper in some means. ■

If we choose another Lyapunov–Krasovskii functional, then we have the following sufficient conditions to guarantee the global asymptotic stability of the origin of model (19).

Theorem 3.4: Suppose that $0 \leq \dot{\tau}_i(t) \leq \eta_i < 1$. Then, the origin of system (19) is globally asymptotically stable if there exist positive–definite symmetric matrices $Q_i > 0, M_i > 0$, and $P > 0$, a positive diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$ such that (31), shown at the bottom of the next page, holds, where $\Theta_i = -(1-\eta_i)Q_i$, $\Xi = -2DA\Delta^{-1} + DW + W^T D + \sum_{i=1}^{N} Q_i$, and $\Omega_i = -(1-\eta_i)M_i, i = 1, \ldots, N$.

Proof: Consider the following Lyapunov–Krasovskii functional:

$$V(x(t)) = x^T(t)Px(t) + 2\sum_{i=1}^{n} d_i \int_0^{x_i(t)} f_i(s)ds + \sum_{i=1}^{N}\int_{t-\tau_i(t)}^{t}(f^T(x(s))Q_i f(x(s)) + x^T(s)M_i x(s))ds. \quad (32)$$

The derivative of (32) along the trajectories of (19) is as follows:

$$\begin{aligned}\dot V(x(t)) \le & -2x^T(t)PAx(t) + 2x^T(t)PWf(x(t)) \\ & + 2x^T(t)P\sum_{i=1}^{N} W_i f(x(t-\tau_i(t))) \\ & - 2f^T(x(t))DA\Delta^{-1}f(x(t)) \\ & + 2f^T(x(t))DWf(x(t)) \\ & + 2f^T(x(t))D\sum_{i=1}^{N} W_i f(x(t-\tau_i(t))) \\ & + \sum_{i=1}^{N}(f^T(x(t))Q_i f(x(t)) \\ & - (1-\eta_i)f^T(x(t-\tau_i(t)))Q_i f(x(t-\tau_i(t)))) \\ & + \sum_{i=1}^{N}(x^T(t)M_i x(t) \\ & - (1-\eta_i)x^T(t-\tau_i(t))M_i x(t-\tau_i(t))) \\ = & \ v^T(t)\Psi v(t)\end{aligned}$$

where $v(t) = (x^T(t)\ f^T(x(t))\ f^T(x(t-\tau_1(t)))\cdots f^T(x(t-\tau_N(t)))\ x^T(t-\tau_1(t))\cdots x^T(t-\tau_N(t)))^T$, and we have used $-f^T(x(t))DAx(t) \le -f^T(x(t))DA\Delta^{-1}f(x(t))$. Then, $\dot V(x(t)) \le v^T(t)\Psi v(t) < 0$ for $v(t) \ne 0$, where Ψ is defined in (31). By Lyapunov stability theory, the origin of system (19) is globally asymptotically stable.

Corollary 3.4: Suppose that $0 \le \dot\tau_i(t) \le \eta_i < 1$. Then, the origin of system (19) is globally asymptotically stable if there exist positive–definite symmetric matrices $Q_i > 0$, $M_i > 0$, and $P > 0$, a positive diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$, and positive–semidefinite diagonal matrices M and R_i, such that (33), shown at the bottom of the page, holds, where $\Theta_i = -(1-\eta_i)Q_i - 2R_i$, $\Upsilon = -2M + DW + W^TD + \sum_{i=1}^{N} Q_i$, and $\Omega_i = -(1-\eta_i)M_i, i = 1, \ldots, N$.

Proof: Consider the Lyapunov–Krasovskii functional defined in (32). The derivative of (32) along the trajectories of (19) is as follows:

$$\begin{aligned}\dot V(x(t)) \le & -2x^T(t)PAx(t) + 2x^T(t)PWf(x(t)) \\ & + 2x^T(t)P\sum_{i=1}^{N} W_i f(x(t-\tau_i(t))) \\ & - 2f^T(x(t))DAx(t) + 2f^T(x(t))DWf(x(t)) \\ & + 2f^T(x(t))D\sum_{i=1}^{N} W_i f(x(t-\tau_i(t))) \\ & + \sum_{i=1}^{N}(f^T(x(t))Q_i f(x(t)) \\ & - (1-\eta_i)f^T(x(t-\tau_i(t)))Q_i f(x(t-\tau_i(t)))) \\ & + \sum_{i=1}^{N}(x^T(t)M_i x(t) \\ & - (1-\eta_i)x(t-\tau_i(t))^T M_i x(t-\tau_i(t))) \\ & + 2\sum_{i=1}^{N} f^T(x(t-\tau_i(t)))R_i\Delta x(t-\tau_i(t)) \\ & - 2\sum_{i=1}^{N} f^T(x(t-\tau_i(t)))R_i f(x(t-\tau_i(t))) \\ & + 2f^T(x(t))M\Delta x(t) - 2f^T(x(t))Mf(x(t)) \\ = & \ v^T(t)\Psi_1 v(t) \quad (34)\end{aligned}$$

$$\Psi = \begin{bmatrix} -PA - AP + \sum_{i=1}^{N} M_i & PW & PW_1 & \cdots & PW_N & 0 & \cdots & 0 \\ W^TP & \Xi & DW_1 & \cdots & DW_N & 0 & \cdots & 0 \\ W_1^TP & W_1^TD & \Theta_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & 0 & \cdots & 0 \\ W_N^TP & W_N^TD & 0 & \cdots & \Theta_N & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & \Omega_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \Omega_N \end{bmatrix} < 0 \quad (31)$$

$$\Psi_1 = \begin{bmatrix} -PA - AP + \sum_{i=1}^{N} M_i & PW - DA + M\Delta & PW_1 & \cdots & PW_N & 0 & \cdots & 0 \\ W^TP - AD + M\Delta & \Upsilon & DW_1 & \cdots & DW_N & 0 & \cdots & 0 \\ W_1^TP & W_1^TD & \Theta_1 & \cdots & 0 & R_1\Delta & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ W_N^TP & W_N^TD & 0 & \cdots & \Theta_N & 0 & \cdots & R_N\Delta \\ 0 & 0 & R_1\Delta & \cdots & 0 & \Omega_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & R_N\Delta & 0 & \cdots & \Omega_N \end{bmatrix} < 0 \quad (33)$$

where we have used

$$f^T(x(t-\tau_i(t)))R_i f(x(t-\tau_i(t))) \le f^T(x(t-\tau_i(t)))R_i\Delta x(t-\tau_i(t)), \qquad i=1,\dots,N$$

and $2f^T(x(t))Mf(x(t)) \le 2f^T(x(t))M\Delta x(t)$, and $v(t)$ is defined the same way as that in Theorem 3.4. Then, $\dot V(x(t)) \le v^T\Psi_1 v(t) < 0$ for $v(t) \ne 0$, where Ψ_1 is defined in (33). By Lyapunov stability theory, the origin of system (19) is globally asymptotically stable. ∎

Remark 3.9: From the proof of Corollary 3.4, we can derive another expression for $\dot V(x(t))$. From (34), we have

$$\begin{aligned}\dot V(x(t)) &\le v^T(t)\Psi_1 v(t) + 2f^T(x(t))DA\Delta^{-1}f(x(t))\\ &\quad - 2f^T(x(t))DA\Delta^{-1}f(x(t))\\ &= v^T(t)\Psi v(t) + 2f^T(x(t))DA\Delta^{-1}f(x(t))\\ &\quad - 2f^T(x(t))DAx(t)\\ &\quad + 2\sum_{i=1}^N f^T(x(t-\tau_i(t)))R_i\Delta x(t-\tau_i(t))\\ &\quad - 2\sum_{i=1}^N f^T(x(t-\tau_i(t)))R_i f(x(t-\tau_i(t)))\\ &\quad + 2f^T(x(t))M\Delta x(t) - 2f^T(x(t))Mf(x(t)) \qquad (35)\end{aligned}$$

where Ψ and Ψ_1 are defined in (31) and (33), respectively. Now, we have two cases to discuss for (35).

Case 1) $M\Delta = DA$. In this case, (35) can be written as follows:

$$\begin{aligned}\dot V(x(t)) &\le v^T(t)\Psi_1 v(t) + 2f^T(x(t))DA\Delta^{-1}f(x(t))\\ &\quad - 2f^T(x(t))DA\Delta^{-1}f(x(t))\\ &= v^T(t)\Psi v(t)\\ &\quad + 2\sum_{i=1}^N f^T(x(t-\tau_i(t)))R_i\Delta x(t-\tau_i(t))\\ &\quad - 2\sum_{i=1}^N f^T(x(t-\tau_i(t)))R_i f(x(t-\tau_i(t)))\\ &= v^T(t)\Psi v(t) + \varepsilon_0\end{aligned}$$

where

$$\begin{aligned}0 \le \varepsilon_0 &= 2\sum_{i=1}^N f^T(x(t-\tau_i(t)))R_i\Delta x(t-\tau_i(t))\\ &\quad - 2\sum_{i=1}^N f^T(x(t-\tau_i(t)))R_i f(x(t-\tau_i(t)))\\ &\le \sum_{i=1}^N f^T(x(t-\tau_i(t)))(R_i\Delta\Delta R_i - 2R_i)f(x(t-\tau_i(t)))\\ &\quad + \sum_{i=1}^N x^T(t-\tau_i(t))x(t-\tau_i(t))\\ &= v^T(t)\Omega_0 v(t)\\ &\le \bar\varepsilon_0 v^T(t)v(t)\end{aligned}$$

where we have used

$$2X^TY \le X^TX + Y^TY$$

(a special case of Lemma 2.1) to show that

$$\begin{aligned}&f^T(x(t-\tau_i(t)))R_i\Delta x(t-\tau_i(t))\\ &\quad \le f^T(x(t-\tau_i(t)))R_i\Delta\Delta R_i f(x(t-\tau_i(t)))\\ &\qquad + x^T(t-\tau_i(t))x(t-\tau_i(t))\end{aligned}$$

and $\Omega_0 = \mathrm{diag}(0, 0, R_1\Delta\Delta R_1 - 2R_1, \dots, R_N\Delta\Delta R_N - 2R_N, I_1, \dots, I_N), I_i, i=1,\dots,N$, are identity matrices with compatible dimensions, and $\bar\varepsilon_0 = \lambda_M(\Omega_0)$. Therefore, there exists an $\epsilon_0 \in [0, \bar\varepsilon_0]$ such that $\dot V(x(t)) \le v^T(t)\Psi_1 v(t) = v^T(t)(\Psi + \epsilon_0 I)v(t)$. Obviously, if $\Psi < 0$, the right-hand side of $\Psi_1 = \Psi + \epsilon_0 I$ may not be negative definite. Therefore, in this case, Theorem 3.4 is less conservative than Corollary 3.4.

Case 2) $M\Delta \ne DA$. In this case, (35) can be written as follows:

$$\begin{aligned}\dot V(x(t)) &\le v^T(t)\Psi_1 v(t) + 2f^T(x(t))DA\Delta^{-1}f(x(t))\\ &\quad - 2f^T(x(t))DA\Delta^{-1}f(x(t))\\ &= v^T(t)\Psi v(t) + 2f^T(x(t))DA\Delta^{-1}f(x(t))\\ &\quad - 2f^T(x(t))DAx(t)\\ &\quad + 2\sum_{i=1}^N f^T(x(t-\tau_i(t)))R_i\Delta x(t-\tau_i(t))\\ &\quad - 2\sum_{i=1}^N f^T(x(t-\tau_i(t)))R_i f(x(t-\tau_i(t)))\\ &\quad + 2f^T(x(t))M\Delta x(t) - 2f^T(x(t))Mf(x(t))\\ &\le v^T(t)\Psi v(t) + 2f^T(x(t))M\Delta x(t)\\ &\quad - 2f^T(x(t))Mf(x(t))\\ &\quad + 2\sum_{i=1}^N f^T(x(t-\tau_i(t)))R_i\Delta x(t-\tau_i(t))\\ &\quad - 2\sum_{i=1}^N f^T(x(t-\tau_i(t)))R_i f(x(t-\tau_i(t)))\\ &= v^T(t)\Psi v(t) + \varepsilon_1\end{aligned}$$

where we have used the inequality

$$2f^T(x(t))DAx(t) - 2f^T(x(t))DA\Delta^{-1}f(x(t)) \ge 0$$

and

$$\begin{aligned}0 \le \varepsilon_1 &= 2\sum_{i=1}^N f^T(x(t-\tau_i(t)))R_i\Delta x(t-\tau_i(t))\\ &\quad - 2\sum_{i=1}^N f^T(x(t-\tau_i(t)))R_i f(x(t-\tau_i(t)))\\ &\quad + 2f^T(x(t))M\Delta x(t) - 2f^T(x(t))Mf(x(t))\\ &\le \sum_{i=1}^N f^T(x(t-\tau_i(t)))(R_i\Delta\Delta R_i - 2R_i)f(x(t-\tau_i(t)))\\ &\quad + \sum_{i=1}^N x^T(t-\tau_i(t))x(t-\tau_i(t))\\ &\quad + f^T(x(t))(M\Delta\Delta M - 2M)f(x(t)) + x^T(t)x(t)\\ &= v^T(t)\Omega_c v(t)\\ &\le \bar\varepsilon_1 v^T(t)v(t)\end{aligned}$$

where $\Omega_c = \mathrm{diag}(I, M\Delta\Delta M - 2M, R_1\Delta\Delta R_1 - 2R_1, \ldots, R_N\Delta\Delta R_N - 2R_N, I_1, \ldots, I_N)$, $I_i, i = 1, \ldots, N$, are identity matrices with compatible dimensions, and $\bar{\varepsilon}_1 = \lambda_M(\Omega_c)$. Then, $\dot{V}(x(t)) \leq v^T(t)\Psi_1 v(t) \leq v^T(t)(\Psi + \bar{\varepsilon}_1 I)v(t)$. Obviously, if $\Psi < 0$, the right-hand side of inequality $\Psi_1 \leq \Psi + \bar{\varepsilon}_1 I$ may not be negative definite. Therefore, in this case, Theorem 3.4 is less conservative than Corollary 3.4. ■

Remark 3.10: When $A = I, N = 1$, and $\tau_1(t) = \tau(t)$, model (18) has been studied in [9] and a global asymptotic stability criterion is obtained by constructing a new Lyapunov–Krasovskii functional. The characteristic of this Lyapunov–Krasovskii functional is that an integral term of state $\int_{t-\tau_1(t)}^{t} x^T(s)M_1 x(s)ds$ is involved in the Lyapunov–Krasovskii functional (see [9, eq. (10)]). In [9], it was pointed out that the new Lyapunov–Krasovskii functional together with the S-procedure provided an improved global asymptotic stability criterion. However, the result in [9] can be obtained from the present Corollary 3.4 with $N = 1$ and $A = I$, while Corollary 3.4 is derived only using some inequality calculations. Therefore, S-procedure may not be necessary in the proof of the main result in [9]. ■

Corollary 3.5: Suppose that $0 \leq \dot{\tau}_i(t) \leq \eta_i < 1$. Then, the origin of system (19) is globally asymptotically stable if there exist positive–definite symmetric matrices $Q_i > 0$ and $P > 0$, and positive diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$, such that the following condition holds:

$$\Psi_2 = \begin{bmatrix} -PA - AP & PW & PW_1 & \cdots & PW_N \\ W^TP & \Xi & DW_1 & \cdots & DW_N \\ W_1^TP & W_1^TD & \Theta_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ W_N^TP & W_N^TD & 0 & \cdots & \Theta_N \end{bmatrix} < 0 \quad (36)$$

where $\Theta_i = -(1 - \eta_i)Q_i$ and $\Xi = -2DA\Delta^{-1} + DW + W^TD + \sum_{i=1}^{N} Q_i$, $i = 1, \ldots, N$.

Proof: Consider the following Lyapunov–Krasovskii functional:

$$V(x(t)) = x^T(t)Px(t) + 2\sum_{i=1}^{n} d_i \int_0^{x_i(t)} f_i(s)ds + \sum_{i=1}^{N} \int_{t-\tau_i(t)}^{t} f^T(x(s))Q_i f(x(s))ds. \quad (37)$$

The derivative of (37) along the trajectories of (19) is as follows:

$$\begin{aligned}\dot{V}(x(t)) \leq & -2x^T(t)PAx(t) + 2x^T(t)PWf(x(t)) \\ & + 2x^T(t)P\sum_{i=1}^{N} W_i f(x(t - \tau_i(t))) \\ & - 2f^T(x(t))DA\Delta^{-1}f(x(t)) \\ & + 2f^T(x(t))DWf(x(t)) \\ & + 2f^T(x(t))D\sum_{i=1}^{N} W_i f(x(t - \tau_i(t))) \\ & + \sum_{i=1}^{N}(f^T(x(t))Q_i f(x(t)) \\ & - (1 - \eta_i)f^T(x(t - \tau_i(t)))Q_i f(x(t - \tau_i(t))))\end{aligned}$$

where we have used

$$-2f^T(x(t))Dx(t) \leq -2f^T(x(t))D\Delta^{-1}f(x(t)).$$

Let $v(t) = [x^T(t)\ f^T(x(t))\ f^T(x(t - \tau_1(t)))\ \cdots\ f^T(x(t - \tau_N(t)))]^T$, then $\dot{V}(x(t)) \leq v^T\Psi_2 v(t) < 0$ for $v(t) \neq 0$. By Lyapunov stability theory, the origin of system (19) is globally asymptotically stable. ■

Remark 3.11: When $A = I$, $N = 1$, $\tau_1(t) = \tau$, and $g_i(u_i(t)) = 0.5(|u_i(t) + 1| - |u_i(t) - 1|), i = 1, \ldots, n$, model (18) has been studied in [27]. Theorem 1 in [27] can be restated to require that

$$\Psi_2 = \begin{bmatrix} -2P & PW & PW_1 \\ W^TP & -2D + DW + W^TD + Q_1 & DW_1 \\ W_1^TP & W_1^TD & -Q_1 \end{bmatrix} < 0 \quad (38)$$

which is just Corollary 3.5 with $A = I$, $\Delta = I$, and $N = 1$. However, as shown in [24], (38) is essentially equivalent to

$$-2D + DW + W^TD + Q_1 + DW_1Q_1^{-1}W_1^TD < 0$$

which is the main result in [24]. Therefore, the result in [27] is a special case of Theorem 3.3. Similarly, Theorems 1 and 2 in [4] are also special cases of the present Corollary 3.5 and Theorem 3.3. Moreover, if we let $N = 1, A = \Delta = I$, and $M_1 = \epsilon I$ in Theorem 3.4 or let $N = 1$, $A = \Delta = I, M\Delta = DA, R_1 = 0$, and $M_1 = \epsilon I$ in Corollary 3.4, where $\epsilon > 0$ is a sufficiently small scalar, then using the same approach as in [24], Theorem 1 in [27] can also be derived from Theorem 3.4 and Corollary 3.4. ■

Remark 3.12: Similar to the proof in [24], we can directly see that Corollary 3.5 is equivalent to Theorem 3.3. Similarly, Theorem 3.4 is equivalent to Corollary 3.5. Therefore, the present Theorem 3.3, Theorem 3.4, and Corollary 3.5 are equivalent in essence and are less conservative than the results of [4], [24], and [27]. ■

Remark 3.13: Now, we make a comparison among Theorems 3.1–3.4. Obviously, from the viewpoint of different multiple time delays, model (18) is a special case of model (7) or (8). From the viewpoint of different activation functions, model (18) and model (7) or (8) are special cases of model (1) or (2). Therefore, Theorem 3.1 is more general than Theorem 3.2. Both Theorems 3.1 and 3.2 are more general than Theorems 3.3 and 3.4, and have wider application domains than Theorems 3.3 and 3.4. To the best of our knowledge, no stability results in the form of LMI are reported in the literature for neural network model (1) or (2), even for model (7) or (8), while this paper established such LMI-based stability results for both model (2) and model (8). It is important to establish the LMI-based stability results for model (2) and model (8) as well as for model (18). Although model (16), as a special case of model (7), has been studied in the literature, for instance, [11], [14], and [21], all the results in [11], [14], and [21] did not consider the neuron's excitatory and inhibitory effects on neural networks, which led to stability results that are more conservative. ■

Remark 3.14: When $\dot{\tau}_{ij}(t) \leq 0$, both condition (3) in Proposition 3.1 and condition (9) in Proposition 3.2 should be modified as follows. We can let γ_i in Propositions 3.1 and 3.2 be

1 and derive similar condition as in (3) and (9), respectively. Similarly, for the case of $\dot{\tau}_i(t) \leq 0$, we can also obtain the corresponding results if we replace $1-\eta_i$ by 1 in Theorems 3.3 and 3.4 and Corollaries 3.3–3.5, respectively. ■

Remark 3.15: If the activation functions $g_j(\cdot)$ and $f_j(\cdot)$ in Assumption 2.1 are bounded, then both neural networks (1) and (7) have an equilibrium point [5], [18], [30]. In this case, the condition $0 \leq \dot{\tau}_{ij}(t) \leq \mu_{ij} < 1$ in Proposition 3.1, Theorem 3.1, Proposition 3.2, and Theorem 3.2 can be relaxed to $\dot{\tau}_{ij}(t) \leq \mu_{ij} < 1$. Similarly, the condition $0 \leq \dot{\tau}_i(t) \leq \eta_i < 1$ in Theorems 3.3 and 3.4 and Corollaries 3.3–3.5 can be relaxed to $\dot{\tau}_i(t) \leq \eta_i < 1$, respectively. ■

C. Recurrent Neural Networks With Constant Delays

Now, consider the system (19) with constant delays, which requires the use of a different Lyapunov–Krasovskii functional.

Theorem 3.5 (See Appendix VI for a Proof): The origin of system (19) with constant delays is globally asymptotically stable if

$$-\Delta^{-2}-2\Delta^{-1}+A^{-1}W+W^TA^{-1}+(N+1)W^TA^{-1}A^{-1}W + \sum_{i=1}^{N}\left[(N+1)W_i^TA^{-1}A^{-1}W_i+I+A^{-1}W_iW_i^TA^{-1}\right]<0. \tag{39}$$

For neural networks with constant delays described by

$$\dot{x}(t) = -Ax(t) + \sum_{i=1}^{N} W_i f(x(t-\tau_i)) \tag{40}$$

we have the following stability result.

Corollary 3.6: The origin of system (40) is globally asymptotically stable if

$$-\Delta^{-2} - 2\Delta^{-1} + \sum_{i=1}^{N}\Big(NW_i^TA^{-1}A^{-1}W_i + I + A^{-1}W_iW_i^TA^{-1}\Big) < 0. \tag{41}$$

Proof: Choose the following Lyapunov–Krasovskii functional:

$$\begin{aligned} V(x(t)) &= x^T(t)A^{-1}x(t) \\ &\quad + \sum_{i=1}^{N}\int_{t-\tau_i}^{t} f^T(x(s))P_if(x(s))ds \\ &\quad + \sum_{i=1}^{n}\frac{2}{a_i}\int_0^{x_i(t)} f_i(s)ds \end{aligned}$$

which is the same as that in the proof of Theorem 3.5 [see (93) in Appendix VI]. Choose $P_i = N(W_i^TA^{-1}A^{-1}W_i) + I, i = 1,\ldots,N$. By $-2x^T(t)x(t) = -N(1/N)x^T(t)x(t) - x^T(t)x(t)$, and substituting (96) in Appendix VI by the following inequalities:

$$\begin{aligned} -\frac{1}{N}x^T(t)x(t) &+ 2x^T(t)A^{-1}W_if(x(t-\tau_i)) \\ &\leq Nf^T(x(t-\tau_i))W_i^TA^{-1}A^{-1}W_if(x(t-\tau_i)) \end{aligned}$$

we can prove Corollary 3.6 similarly to the Proof of Theorem 3.5. The details are omitted. ■

Remark 3.16: When $N = 1$ in (40), the asymptotic stability of the equilibrium point was studied in [1]. Theorem 2 in [1] requires

$$-\Delta^{-2} - 2\Delta^{-1} + 2A^{-1}W_1W_1^TA^{-1} + I < 0 \tag{42}$$

while in this case, our result, Corollary 3.6 with $N = 1$, requires

$$-\Delta^{-2}-2\Delta^{-1}+A^{-1}W_1W_1^TA^{-1}+W_1^TA^{-1}A^{-1}W_1+I<0. \tag{43}$$

Obviously, (42) and (43) are not equivalent. The reason for this is that Theorem 2 in [1] can only be applied when $A^{-1}W_1 = W_1^TA^{-1}$ or when W_1 is a diagonal matrix, under which the previous conditions (42) and (43) are equivalent (see the equation before (8) in [1]). ■

IV. Application Example

Consider the continuous PH neutralization of an acid stream by a highly concentrated basic stream, which can be expressed in the following form [23]:

$$v\dot{y}(t) = -fy(t) - u(t), \quad PH = w_2\tanh(w_1y(t)) \tag{44}$$

where v is the volume of the mixing tank, $u(t)$ is the manipulated variable representing the base flow rate, f is the acid flow rate, w_1 and w_2 are some constants, $y(t)$ is the strong acid equivalent, and PH is the measured output signal.

In fact, time delay is always inevitable in the control process. Therefore, we slightly modify model (44) as follows:

$$\begin{cases} v\dot{y}(t) = -fy(t) - u(t), \\ PH = w_2\tanh(w_1y(t)) + w_2w_3\tanh(w_1y(t-\tau)). \end{cases} \tag{45}$$

Adopting output feedback controller $u = -K * PH$, the purpose is to design an appropriate feedback gain K such that the closed-loop system is asymptotically stable. The closed-loop system can be expressed in the following form:

$$\dot{y}(t) = -\frac{f}{v}y(t) + \frac{Kw_2}{v}\tanh(w_1y(t)) + \frac{Kw_2w_3}{v}\tanh(w_1y(t-\tau)). \tag{46}$$

Let $x(t) = w_1y(t)$. Then, system (46) is changed to the following form:

$$\dot{x}(t) = -Ax(t) + W\tanh(x(t)) + W_1\tanh(x(t-\tau)) \tag{47}$$

where $A = -f/v$, $B = w_1w_2/v$, $W = BK$, $B_1 = w_1w_2w_3/v$,, and $W_1 = B_1K$. Obviously, the origin is the equilibrium point of system (47), and K is to be designed.

The closed-loop system (47) has the same structure as system (18). Applying Theorem 3.3 to system (47), we require

$$\begin{bmatrix} -2DA\Delta^{-1} + DBK + (DBK)^T + P_1 & DB_1K \\ (DB_1K)^T & -P_1 \end{bmatrix} < 0. \tag{48}$$

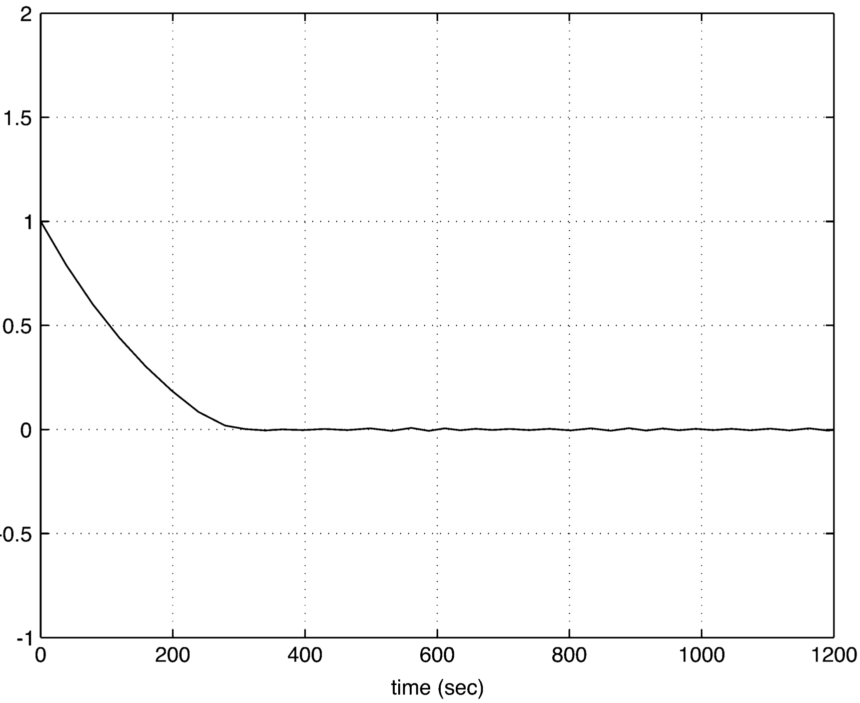

Fig. 1. Response curve of closed-loop system (46) with $K = 0.5022$.

Pre- and postmultiplying $\mathrm{diag}(D^{-1}, D^{-1})$ on both sides of (48), we have

$$\begin{bmatrix} \Lambda & B_1KD^{-1} \\ D^{-1}K^TB_1^T & -D^{-1}P_1D^{-1} \end{bmatrix} < 0 \tag{49}$$

where $\Lambda = -2A\Delta^{-1}D^{-1} + D^{-1}K^TB^T + BKD^{-1} + D^{-1}P_1D^{-1}$. Let $D^{-1} = X$, $KD^{-1} = D_K$, and $D^{-1}P_1D^{-1} = D_P$. Then, (49) becomes

$$\begin{bmatrix} -2A\Delta^{-1}X + D_K^TB^T + BD_K + D_P & B_1D_K \\ D_K^TB_1^T & -D_P \end{bmatrix} < 0. \tag{50}$$

If (50) has solutions X, D_K, and D_P, then feedback gain $K = D_KX^{-1}$.

We take $f = 5.8154, v = 1500.3732, w_1 = 28.9860, w_2 = -3.8500, w_3 = 0.56,$, and $\tau = 1$. Then, $A = 0.0039, B = -0.0744$, and $B_1 = -0.0417$. Solving inequality (50), we have $D_K = 1.5067, D_P = 0.1716, X = 3$, and $K = 0.5022$. For the closed-loop system (47) with previous parameters, Theorems 1 and 3 in [6] are not satisfied.

The response curve of closed-loop system (46) is depicted in Fig. 1.

Now, we consider model (44), which can directly be derived from model (45) if we let $w_3 = 0$. In this case, solving inequality (50), we have $D_K = 8.4185, D_P = 0.9052, X = 3$, and the feedback gain $K = 2.8056$.

The response curve of closed-loop system (46) with $w_3 = 0$ is depicted in Fig. 2.

V. Conclusion

Several sufficient conditions are established for the global asymptotic stability of recurrent neural networks with different multiple time-varying delays. The Lyapunov–Krasovskii stability theory for functional differential equations and the LMI approach are employed in this study. Global asymptotic stability results are also established for recurrent neural networks with multiple time-varying delays. These results are shown to be generalizations of some previously published results and are less conservative than existing results. Finally, global asymptotic stability results are established for recurrent neural networks with constant time delays.

Appendix I
Proof of Proposition 3.1

The primary goal of Proposition 3.1 is to find the condition to ensure that recurrent neural networks (2) has a unique equilibrium point. An equilibrium point $u^\star$ is a constant solution of (2), i.e., it satisfies the algebraic equation

$$-Au^\star + W\bar{g}(u^\star) + \sum_{i=1}^{N} W_i\bar{f}(u^\star) + U = 0. \tag{51}$$

Similar to the proof in [7] and [30], we define the following map associated with (51):

$$H(u) = -Au + W\bar{g}(u) + \sum_{i=1}^{N} W_i\bar{f}(u) + U. \tag{52}$$

Now, we prove that $H(u)$ is a homeomorphism of $\Re^n$ by two steps.

First, we will show that $H(u)$ is injective on $\Re^n$. Suppose, for purpose of contradiction, that there exists two vectors $u, v \in \Re^n$ with $u \neq v$ such that $H(u) = H(v)$. Then

$$0 = -A(u-v) + W[\bar{g}(u) - \bar{g}(v)] + \sum_{i=1}^{N} W_i[\bar{f}(u) - \bar{f}(v)]. \tag{53}$$

Because $u \neq v$, then $\bar{g}(u)-\bar{g}(v) \neq 0$ and $\bar{f}(u)-\bar{f}(v) \neq 0$. Premultiplying $2(u-v)^T P$, $2(\bar{g}(u)-\bar{g}(v))^T H$, and $2(\bar{f}(u)-\bar{f}(v))^T M$ on both sides of (53) yields

$$\begin{aligned} 0 = &-2(u-v)^T PA(u-v) + 2(u-v)^T PW(\bar{g}(u)-\bar{g}(v)) \\ &+ 2(u-v)^T P \sum_{i=1}^{N} W_i(\bar{f}(u)-\bar{f}(v)) \\ &- 2(\bar{g}(u)-\bar{g}(v))^T HA(u-v) \\ &+ 2(\bar{g}(u)-\bar{g}(v))^T HW(\bar{g}(u)-\bar{g}(v)) \\ &+ 2(\bar{g}(u)-\bar{g}(v))^T H \sum_{i=1}^{N} W_i(\bar{f}(u)-\bar{f}(v)) \\ &- 2(\bar{f}(u)-\bar{f}(v))^T MA(u-v) \\ &+ 2(\bar{f}(u)-\bar{f}(v))^T MW(\bar{g}(u)-\bar{g}(v)) \\ &+ 2(\bar{f}(u)-\bar{f}(v))^T M \sum_{i=1}^{N} W_i(\bar{f}(u)-\bar{f}(v)). \end{aligned} \tag{54}$$

By Assumption 2.1, the following inequalities hold:

$$\begin{aligned} &2(\bar{g}(u)-\bar{g}(v))^T H_g \Delta(u-v) \\ &\quad - 2(\bar{g}(u)-\bar{g}(v))^T H_g(\bar{g}(u)-\bar{g}(v)) \geq 0 \end{aligned} \tag{55}$$

$$\begin{aligned} &2(\bar{f}(u)-\bar{f}(v))^T M_f L(u-v) \\ &\quad - 2(\bar{f}(u)-\bar{f}(v))^T M_f(\bar{f}(u)-\bar{f}(v)) \geq 0. \end{aligned} \tag{56}$$

Noting that $0 < \gamma_i \leq 1$, then

$$\begin{aligned} &(\bar{f}(u)-\bar{f}(v))^T Q_i(\bar{f}(u)-\bar{f}(v)) \\ &\quad -\gamma_i(\bar{f}(u)-\bar{f}(v))^T Q_i(\bar{f}(u)-\bar{f}(v)) \geq 0 \end{aligned} \tag{57}$$

for any positive diagonal matrix $Q_i, i=1,\ldots,N$.

Substituting (55)–(57) into (54), we have

$$\begin{aligned} 0 \leq &-2(u-v)^T PA(u-v) \\ &+ 2(u-v)^T PW(\bar{g}(u)-\bar{g}(v)) \\ &+ 2(u-v)^T P \sum_{i=1}^{N} W_i(\bar{f}(u)-\bar{f}(v)) \\ &- 2(\bar{g}(u)-\bar{g}(v))^T HA(u-v) \\ &+ 2(\bar{g}(u)-\bar{g}(v))^T HW(\bar{g}(u)-\bar{g}(v)) \\ &+ 2(\bar{g}(u)-\bar{g}(v))^T H \sum_{i=1}^{N} W_i(\bar{f}(u)-\bar{f}(v)) \\ &- 2(\bar{f}(u)-\bar{f}(v))^T MA(u-v) \\ &+ 2(\bar{f}(u)-\bar{f}(v))^T MW(\bar{g}(u)-\bar{g}(v)) \\ &+ 2(\bar{f}(u)-\bar{f}(v))^T M \sum_{i=1}^{N} W_i(\bar{f}(u)-\bar{f}(v)) \\ &+ 2(\bar{g}(u)-\bar{g}(v))^T H_g \Delta(u-v) \\ &- 2(\bar{g}(u)-\bar{g}(v))^T H_g(\bar{g}(u)-\bar{g}(v)) \\ &+ 2(\bar{f}(u)-\bar{f}(v))^T M_f L(u-v) \\ &- 2(\bar{f}(u)-\bar{f}(v))^T M_f(\bar{f}(u)-\bar{f}(v)) \\ &+ \sum_{i=1}^{N}(\bar{f}(u)-\bar{f}(v))^T Q_i(\bar{f}(u)-\bar{f}(v)) \\ &- \sum_{i=1}^{N} \gamma_i(\bar{f}(u)-\bar{f}(v))^T Q_i(\bar{f}(u)-\bar{f}(v)) \\ \leq &\ \xi^T \Xi_w \xi \end{aligned} \tag{58}$$

where

$$\xi^T = [(u-v)^T, (\bar{g}(u)-\bar{g}(v))^T, (\bar{f}(u)-\bar{f}(v))^T, \ldots, (\bar{f}(u)-\bar{f}(v))^T]$$

and Ξ_w is the same as in (3).

From (3), we have $\xi^T \Xi_w \xi < 0$ for any $\xi \neq 0$. Obviously, this contradicts with (58). Therefore, $u=v$ and $H(u)$ is injective.

Second, we will show that $\|H(u)\|$ approaches infinity as $\|u\|$ approaches infinity. If $\bar{g}(u)$ and $\bar{f}(u)$ are bounded as $\|u\|$ approaches infinity, it is easy to verify that when $\|u\| \to \infty$, $\|H(u)\| \to \infty$. For the case that $\bar{g}(u)$ and $\bar{f}(u)$ are unbounded, we will show that $\|H(u)\| \to \infty$ as $\|u\| \to \infty$. Let

$$\tilde{H}(u) = -Au + W\tilde{g}(u) + \sum_{i=1}^{N} W_i \tilde{f}(u) \tag{59}$$

where $\tilde{g}(u) = \bar{g}(u) - \bar{g}(0)$ and $\tilde{f}(u) = \bar{f}(u) - \bar{f}(0)$. Obviously, $\|\tilde{H}(u)\| \to \infty$ is equivalent to $\|H(u)\| \to \infty$.

Multiplying $2(u^T P + \tilde{g}^T(u)H + \tilde{f}^T(u)M)$ on both sides of (59), and by Assumption 2.1, we have

$$\begin{aligned} &2(u^T P + \tilde{g}^T(u)H + \tilde{f}^T(u)M)\tilde{H}(u) \\ &= 2u^T P \left[-Au + W\tilde{g}(u) + \sum_{i=1}^{N} W_i \tilde{f}(u)\right] \\ &\quad + 2\tilde{g}^T(u)H \left[-Au + W\tilde{g}(u) + \sum_{i=1}^{N} W_i \tilde{f}(u)\right] \\ &\quad + 2\tilde{f}^T(u)M \left[-Au + W\tilde{g}(u) + \sum_{i=1}^{N} W_i \tilde{f}(u)\right] \\ &\leq 2u^T P \left[-Au + W\tilde{g}(u) + \sum_{i=1}^{N} W_i \tilde{f}(u)\right] \\ &\quad + 2\tilde{g}^T(u)H \left[-Au + W\tilde{g}(u) + \sum_{i=1}^{N} W_i \tilde{f}(u)\right] \\ &\quad + 2\tilde{f}^T(u)M \left[-Au + W\tilde{g}(u) + \sum_{i=1}^{N} W_i \tilde{f}(u)\right] \\ &\quad + 2\tilde{g}^T(u)H_g \Delta u - 2\tilde{g}^T(u)H_g \tilde{g}(u) \\ &\quad + 2\tilde{f}^T(u)M_f Lu - 2\tilde{f}^T(u)M_f \tilde{f}(u) \\ &\quad + \sum_{i=1}^{N} \tilde{f}^T(u)Q_i \tilde{f}(u) - \sum_{i=1}^{N} \tilde{f}^T(u)\gamma_i Q_i \tilde{f}(u) \\ &= \bar{\xi}^T \Xi_w \bar{\xi} \end{aligned} \tag{60}$$

where $\bar{\xi}^T = [u^T, \tilde{g}(u)^T, \tilde{f}(u)^T, \ldots, \tilde{f}(u)^T]$ and Ξ_w is the same as in (3).

By (3), there exists a sufficiently small constant $\varepsilon_g > 0$ such that $\bar{\xi}^T \Xi_w \bar{\xi} \leq -\varepsilon_g \bar{\xi}^T \bar{\xi}$. Note that $\bar{\xi}^T \bar{\xi} \geq [u^T, \tilde{g}(u)^T, \tilde{f}(u)^T][u^T, \tilde{g}(u)^T, \tilde{f}(u)^T]^T = \underline{\xi}^T \underline{\xi}$, where $\underline{\xi}^T = [u^T, \tilde{g}(u)^T, \tilde{f}(u)^T]$. Then, from (60), we have

$$\begin{aligned} &2(u^T P + \tilde{g}^T(u)H + \tilde{f}^T(u)M)\tilde{H}(u) \\ &\quad \leq \bar{\xi}^T \Xi_w \bar{\xi} \leq -\varepsilon_g \bar{\xi}^T \bar{\xi} \leq -\varepsilon_g \underline{\xi}^T \underline{\xi} < 0. \end{aligned} \tag{61}$$

From (61), we have

$$\|2(u^T P + \tilde{g}^T(u)H + \tilde{f}^T(u)M)\tilde{H}(u)\| = \|\underline{\xi}^T P_H \tilde{H}(u)\| \geq \varepsilon_g \underline{\xi}^T \underline{\xi}$$

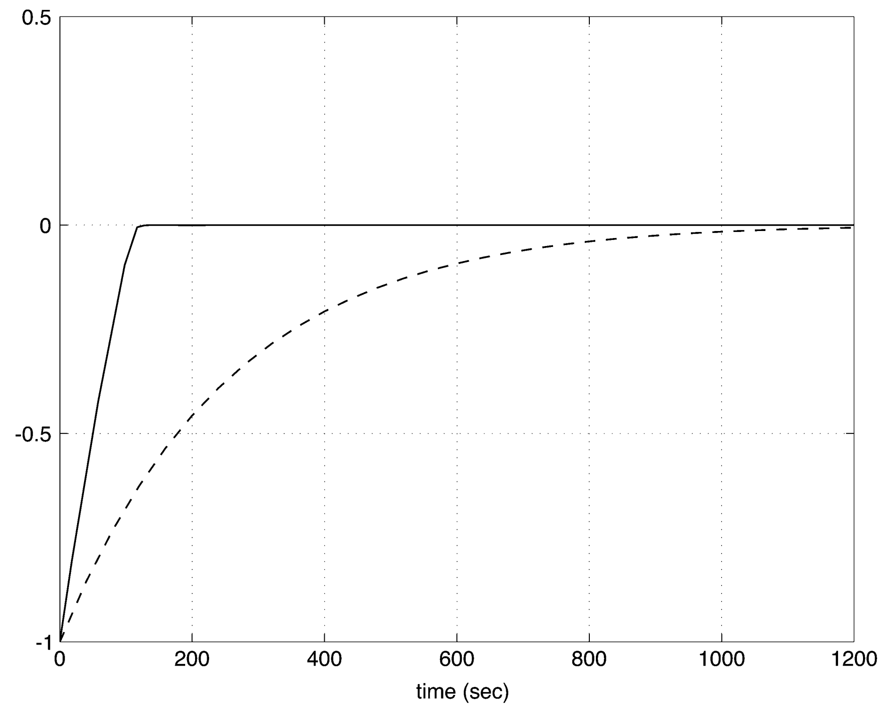

Fig. 2. Response curve of closed-loop system (46) with $w_3 = 0$. Solid line is with $K = 2.8056$; dashed line is the result of [23] with $K = 0.0091$.

where

$$P_H = 2\begin{bmatrix} P \\ H \\ M \end{bmatrix}.$$

Therefore, $\|P_H\|\|\tilde{H}(u)\| \geq \varepsilon_g\|\underline{\xi}\|$. Clearly, $\|\tilde{H}(u)\| \to \infty$ as $\|\underline{\xi}\| \to \infty$, which is equivalent to $\|H(u)\| \to \infty$ as $\|u\| \to \infty$.

By Lemma 2.2, $H(u)$ is a homeomorphism of $\Re^n$. Therefore, for every external constant input U, system (2) has a unique equilibrium point $u^\star$. This completes the proof of Proposition 3.1. ■

APPENDIX II
PROOF OF THEOREM 3.1

Consider the following Lyapunov–Krasovskii functional for model (4):

$$\begin{aligned} V(x(t)) = {} & x^T(t)Px(t) + 2\sum_{i=1}^{n} h_i \int_0^{x_i(t)} g_i(s)ds \\ & + 2\sum_{i=1}^{n} m_i \int_0^{x_i(t)} f_i(s)ds \\ & + \sum_{i=1}^{N}\sum_{j=1}^{n} \int_{t-\tau_{ij}(t)}^{t} q_{ij} f_j^2(x_j(s))ds \end{aligned} \tag{62}$$

where $h_j > 0, m_j > 0$ and $q_{ij} > 0, j = 1,\ldots,n, i = 1,\ldots,N$. The time derivative of the functional (62) along the trajectories of model (4) is obtained as follows:

$$\begin{aligned} & \dot{V}(x(t)) \\ & \leq 2x^T P\left[-Ax(t) + Wg(x(t)) + \sum_{i=1}^{N} W_i f(x(t-\bar{\tau}_i(t)))\right] \\ & \quad + 2g^T(x(t))H\left[-Ax(t) + Wg(x(t)) + \sum_{i=1}^{N} W_i f(x(t-\bar{\tau}_i(t)))\right] \\ & \quad + 2f^T(x(t))M\left[-Ax(t) + Wg(x(t)) + \sum_{i=1}^{N} W_i f(x(t-\bar{\tau}_i(t)))\right] \\ & \quad + \sum_{i=1}^{N}\big[f^T(x(t))Q_i f(x(t)) \\ & \qquad - \gamma_i f^T(x(t-\bar{\tau}_i(t)))Q_i f(x(t-\bar{\tau}_i(t)))\big]. \end{aligned} \tag{63}$$

By Assumption 2.1, we have

$$2f^T(x(t))M_f Lx(t) - 2f^T(x(t))M_f f(x(t)) \geq 0 \tag{64}$$

$$2g^T(x(t))H_g \Delta x(t) - 2g^T(x(t))H_g g(x(t)) \geq 0 \tag{65}$$

where M_f and H_g are positive diagonal matrices.

Substituting (64) and (65) into (63) yields

$$\dot{V}(x(t)) \leq \zeta^T \Xi_w \zeta < 0 \tag{66}$$

for $\zeta \neq 0$, where $\zeta^T = [x^T(t), g^T(x(t)), f^T(x(t)), f^T(x(t-\bar{\tau}_1(t))), f^T(x(t-\bar{\tau}_2(t))), \ldots, f^T(x(t-\bar{\tau}_N(t)))]$ and Ξ_w is the same as in (3).

Thus, by Lyapunov stability theory, it follows that the origin of system (4) or the equilibrium point of system (2) is globally asymptotically stable, independent of the magnitude of time-varying delays. This completes the proof of Theorem 3.1. ■

APPENDIX III
PROOF OF PROPOSITION 3.2

The primary goal of Proposition 3.2 is to find condition ensuring that recurrent neural networks (8) has a unique equilibrium point. An equilibrium point $u^\star$ is a constant solution of (8), i.e., it satisfies the algebraic equation

$$-Au^\star + W\bar{g}(u^\star) + \sum_{i=1}^{N} W_i \bar{g}(u^\star) + U = 0. \quad (67)$$

Similar to the proof in [7] and [30], we define the following map associated with (67):

$$H(u) = -Au + W\bar{g}(u) + \sum_{i=1}^{N} W_i \bar{g}(u) + U. \quad (68)$$

Now, we prove that $H(u)$ is a homeomorphism of $\Re^n$ by two steps.

First, we will show that $H(u)$ is injective on $\Re^n$. Suppose, for purpose of contradiction, that there exists two vectors $u, v \in \Re^n$ with $u \neq v$ such that $H(u) = H(v)$. Then

$$0 = -A(u-v) + W(\bar{g}(u) - \bar{g}(v)) + \sum_{i=1}^{N} W_i(\bar{g}(u) - \bar{g}(v)). \quad (69)$$

Because $u \neq v$, then $\bar{g}(u) - \bar{g}(v) \neq 0$. Multiplying $2(\bar{g}(u) - \bar{g}(v))^T P$ on both sides of (69), and by Assumption 2.1 and Lemma 2.1, yields

$$\begin{aligned}
0 &= -2(\bar{g}(u) - \bar{g}(v))^T PA(u-v) \\
&\quad + 2(\bar{g}(u) - \bar{g}(v))^T PW(\bar{g}(u) - \bar{g}(v)) \\
&\quad + 2(\bar{g}(u) - \bar{g}(v))^T P \sum_{i=1}^{N} W_i(\bar{g}(u) - \bar{g}(v)) \\
&\leq -2(\bar{g}(u) - \bar{g}(v))^T PA\Delta^{-1}(\bar{g}(u) - \bar{g}(v)) \\
&\quad + 2(\bar{g}(u) - \bar{g}(v))^T PW(\bar{g}(u) - \bar{g}(v)) \\
&\quad + \sum_{i=1}^{N} (\bar{g}(u) - \bar{g}(v))^T PW_i\gamma_i^{-1}Q_i^{-1}W_i^T P(\bar{g}(u) - \bar{g}(v)) \\
&\quad + \sum_{i=1}^{N} (\bar{g}(u) - \bar{g}(v))^T Q_i(\bar{g}(u) - \bar{g}(v)) \\
&\leq (\bar{g}(u) - \bar{g}(v))^T \Bigg[-2PA\Delta^{-1} + PW + W^T P \\
&\qquad + \sum_{i=1}^{N} PW_i\gamma_i^{-1}Q_i^{-1}W_i^T P + Q_i \Bigg] \\
&\quad \times (\bar{g}(u) - \bar{g}(v)). \quad (70)
\end{aligned}$$

From (9), we have

$$\begin{aligned}
&(\bar{g}(u) - \bar{g}(v))^T \Bigg[-2PA\Delta^{-1} + PW + W^T P \\
&\qquad + \sum_{i=1}^{N} PW_i\gamma_i^{-1}Q_i^{-1}W_i^T P + Q_i \Bigg] \\
&\times (\bar{g}(u) - \bar{g}(v)) < 0 \quad (71)
\end{aligned}$$

for $\bar{g}(u) - \bar{g}(v) \neq 0$. Obviously, (71) contradicts with (70). Therefore, $u = v$ and $H(u)$ is injective.

Second, we will show that $\|H(u)\|$ approaches infinity as $\|u\|$ approaches infinity. If $\bar{g}(u)$ is bounded, it is easy to verify that when $\|u\| \to \infty$, $\|H(u)\| \to \infty$. For the case that $\bar{g}(u)$ is unbounded, we will show that $\|H(u)\| \to \infty$ as $\|u\| \to \infty$. Let

$$\check{H}(u) = -Au + W\check{g}(u) + \sum_{i=1}^{N} W_i \check{g}(u) \quad (72)$$

where $\check{g}(u) = \bar{g}(u) - \bar{g}(0)$. Obviously, $\|\check{H}(u)\| \to \infty$ is equivalent to $\|H(u)\| \to \infty$.

Multiplying $2\check{g}^T(u)P$ on both sides of (72), and by Assumption 2.1 and Lemma 2.1, we have

$$\begin{aligned}
&2\check{g}^T(u)P\check{H}(u) \\
&\quad = -2\check{g}^T(u)PAu + 2\check{g}^T(u)PW\check{g}(u) + 2\check{g}^T(u)P\sum_{i=1}^{N} W_i\check{g}(u) \\
&\quad \leq \check{g}^T(u) \Bigg[-2PA\Delta^{-1} + PW + W^T P \\
&\qquad + \sum_{i=1}^{N} PW_iQ_i^{-1}\gamma_i^{-1}W_i^T P + Q_i \Bigg] \check{g}(u). \quad (73)
\end{aligned}$$

By (9), there exists a sufficiently small constant $\varepsilon_u > 0$ such that

$$\begin{aligned}
&\check{g}^T(u) \Bigg[-2PA\Delta^{-1} + PW + W^T P \\
&\qquad + \sum_{i=1}^{N} PW_iQ_i^{-1}\gamma_i^{-1}W_i^T P + Q_i \Bigg] \check{g}(u) \\
&\leq -\varepsilon_u \check{g}^T(u)\check{g}(u) \\
&< 0 \quad (74)
\end{aligned}$$

for $\check{g}(u) \neq 0$.

Combining (73) with (74), we have $2\check{g}^T(u)P\check{H}(u) \leq -\varepsilon_u\check{g}^T(u)\check{g}(u)$, or $\|2\check{g}^T(u)P\check{H}(u)\| \geq \varepsilon_u\check{g}^T(u)\check{g}(u)$, or $\|2P\|\|\check{H}(u)\| \geq \varepsilon_u\|\check{g}(u)\|$. Clearly, $\|\check{H}(u)\| \to \infty$ as $\|\check{g}(u)\| \to \infty$, which is equivalent to $\|H(u)\| \to \infty$ as $\|u\| \to \infty$.

By Lemma 2.2, $H(u)$ is a homeomorphism of $\Re^n$. Therefore, for every external constant input U, system (8) have a unique equilibrium point $u^\star$. This completes the proof of Proposition 3.2. ∎

APPENDIX IV
PROOF OF THEOREM 3.2

Consider the following Lyapunov–Krasovskii functional for model (10):

$$\begin{aligned}
V(x(t)) &= (N+1)x^T(t)x(t) + 2\alpha \sum_{i=1}^{n} p_i \int_0^{x_i(t)} f_i(s)ds \\
&\quad + \sum_{i=1}^{N} (\alpha + \beta_i) \sum_{j=1}^{n} \int_{t-\tau_{ij}(t)}^{t} q_{ij} f_j^2(x_j(s))ds \quad (75)
\end{aligned}$$

where $\alpha > 0$ and $\beta_i > 0$ will be defined later, $i = 1, \ldots, N$. The time derivative of the functional (75) along the trajectories of model (10) is obtained as follows:

$$\begin{aligned}\dot{V}(x(t)) \le & -2(N+1)x^T(t)Ax(t) \\ & + 2(N+1)x^T(t)Wf(x(t)) \\ & + 2(N+1)x^T(t)\sum_{i=1}^{N} W_i f(x(t-\bar{\tau}_i(t))) \\ & - 2\alpha f^T(x(t))PAx(t) + 2\alpha f^T(x(t))PWf(x(t)) \\ & + 2\alpha f^T(x(t))P\sum_{i=1}^{N} W_i f(x(t-\bar{\tau}_i(t))) \\ & + \sum_{i=1}^{N}(\alpha+\beta_i)[f^T(x(t))Q_i f(x(t)) \\ & - \gamma_i f^T(x(t-\bar{\tau}_i(t)))Q_i f(x(t-\bar{\tau}_i(t)))]. \end{aligned} \tag{76}$$

By Assumption 2.1, we have

$$-\alpha f^T(x(t))PAx(t) \le -\alpha f^T(x(t))PA\Delta^{-1}f(x(t)). \tag{77}$$

By Lemma 2.1, the following inequality holds:

$$\begin{aligned}-x^T(t)(2A)x(t) + 2(N+1)x^T(t)Wf(x(t)) \\ \le (N+1)^2 f^T(x(t))W^T(2A)^{-1}Wf(x(t)) \end{aligned} \tag{78}$$

and the following inequalities hold for $i = 1, 2, \ldots, N$:

$$\begin{aligned}& -x^T(t)(2A)x(t) + 2(N+1)x^T(t)W_i f(x(t-\bar{\tau}_i(t))) \\ & \quad \le (N+1)^2 f^T(x(t-\bar{\tau}_i(t)))W_i^T(2A)^{-1}W_i f(x(t-\bar{\tau}_i(t))) \end{aligned} \tag{79}$$

$$\begin{aligned}& -\alpha\gamma_i f^T(x(t-\bar{\tau}_i(t)))Q_i f(x(t-\bar{\tau}_i(t))) \\ & \quad + 2\alpha f^T(x(t))PW_i f(x(t-\bar{\tau}_i(t))) \\ & \quad \le \frac{\alpha}{\gamma_i} f^T(x(t))PW_iQ_i^{-1}W_i^TPf(x(t)). \end{aligned} \tag{80}$$

Substituting (77)–(80) into (76) yields

$$\begin{aligned}& \dot{V}(x(t)) \\ & \le f^T(x(t))\Bigg[(N+1)^2W^T(2A)^{-1}W + \sum_{i=1}^{N}\frac{\alpha}{\gamma_i}PW_iQ_i^{-1}W_i^TP \\ & \qquad -2\alpha PA\Delta^{-1} + \sum_{i=1}^{N}(\alpha+\beta_i)Q_i + 2\alpha PW\Bigg]f(x(t)) \\ & \quad + \sum_{i=1}^{N} f^T(x(t-\bar{\tau}_i(t))) \\ & \quad \times \left[(N+1)^2W_i^T(2A)^{-1}W_i - \beta_i\gamma_iQ_i\right] f(x(t-\bar{\tau}_i(t))) \\ & = f^T(x(t))\Bigg[(N+1)^2W^T(2A)^{-1}W + \sum_{i=1}^{N}\beta_iQ_i \\ & \qquad + \alpha\left(\sum_{i=1}^{N}\left(\frac{1}{\gamma_i}PW_iQ_i^{-1}W_i^TP + Q_i\right)\right. \\ & \qquad \left. - 2PA\Delta^{-1} + PW + W^TP\right)\Bigg] f(x(t)) \\ & \quad + \sum_{i=1}^{N} f^T(x(t-\bar{\tau}_i(t))) \\ & \quad \times [(N+1)^2W_i^T(2A)^{-1}W_i - \beta_i\gamma_iQ_i]f(x(t-\bar{\tau}_i(t))) \\ & \le \Bigg[\lambda_M\left((N+1)^2W^T(2A)^{-1}W + \sum_{i=1}^{N}\beta_iQ_i\right) \\ & \qquad - \alpha\lambda_m\left(-\sum_{i=1}^{N}\left(\frac{1}{\gamma_i}PW_iQ_i^{-1}W_i^TP + Q_i\right)\right. \\ & \qquad \left. + 2PA\Delta^{-1} - PW - W^TP\right)\Bigg] \|f(x(t))\|^2 \\ & \quad + \sum_{i=1}^{N} f^T(x(t-\bar{\tau}_i(t))) \\ & \quad \times [(N+1)^2W_i^T(2A)^{-1}W_i - \beta_i\gamma_iQ_i]f(x(t-\bar{\tau}_i(t))). \end{aligned} \tag{81}$$

If we choose $\beta_i > 0, i = 1, 2, \ldots, N$, and α as follows:

$$\begin{aligned}\beta_i & \ge \frac{(N+1)^2\|W_i\|^2\|A^{-1}\|}{2\gamma_i\lambda_m(Q_i)} \\ \alpha & > \frac{\lambda_M\left(\sum_{i=1}^{N}\beta_iQ_i + (N+1)^2W^T(2A)^{-1}W\right)}{\Upsilon}\end{aligned} \tag{82}$$

where

$$\Upsilon = \lambda_m\left(2PA\Delta^{-1} - PW - W^TP - \sum_{i=1}^{N}\left(\frac{1}{\gamma_i}PW_iQ_i^{-1}W_i^TP + Q_i\right)\right)$$

then, from (9) and (81), we can directly obtain $\alpha > 0$ and $\dot{V}(x(t)) < 0$ for any $f(x(t)) \neq 0$.

Note that $f(x(t)) \neq 0$ implies that $x(t) \neq 0$. Now, let $f(x(t)) = 0$ and $x(t) \neq 0$. In this case, $\dot{V}(x(t))$ is in the following form (according to Lemma 2.1):

$$\begin{aligned}\dot{V}(x(t)) \le & -2(N+1)x^T(t)Ax(t) - \sum_{i=1}^{N}(\alpha+\beta_i)\gamma_i f^T \\ & \times (x(t-\bar{\tau}_i(t)))Q_i f(x(t-\bar{\tau}_i(t))) \\ & + 2(N+1)x^T(t)\sum_{i=1}^{N}W_i f(x(t-\bar{\tau}_i(t))) \\ \le & -2(N+1)x^T(t)Ax(t) \\ & + \sum_{i=1}^{N}(N+1)^2x^T(t)W_i[(\alpha+\beta_i)\gamma_iQ_i]^{-1}W_i^Tx(t) \\ = & -2x^T(t)Ax(t) + \sum_{i=1}^{N}x^T(t) \\ & \times ((N+1)^2W_i[(\alpha+\beta_i)\gamma_iQ_i]^{-1}W_i^T - 2A)x(t) \\ \le & -2x^T(t)Ax(t) + \sum_{i=1}^{N}x^T(t) \\ & \times [(N+1)^2W_i(\beta_i\gamma_iQ_i)^{-1}W_i^T - 2A]x(t). \end{aligned}$$

Consider (82) again, then $\dot{V}(x(t)) \leq -2x^T(t)Ax(t) < 0$ for $\forall x(t) \neq 0$.

Now, we consider the case where $f(x(t)) = x(t) = 0$. In this case, $\dot{V}(x(t))$ takes the form

$$\dot{V}(x(t)) \leq -\sum_{i=1}^{N} (\alpha + \beta_i)\gamma_i f^T(x(t-\bar{\tau}_i(t)))Q_i f(x(t-\bar{\tau}_i(t))).$$

It is easily seen that $\dot{V}(x(t)) < 0$ for $\forall f(x(t - \bar{\tau}_i(t))) \neq 0$. Hence, we have proved that $\dot{V}(x(t)) = 0$ if and only if $f(x(t)) = x(t) = f(x(t - \bar{\tau}_i(t))) = 0$. Otherwise, $\dot{V}(x(t)) < 0$. On the other hand, $V(x(t))$ is radially unbounded. Thus, by Lyapunov stability theory, it follows that the origin of system (10) or the equilibrium point of system (8) is globally asymptotically stable, independent of the magnitude of time-varying delays. This completes the proof of Theorem 3.2. ■

APPENDIX V
PROOF OF THEOREM 3.3

Choose a Lyapunov–Krasovskii functional as follows:

$$\begin{aligned} V(x(t)) = {} & x^T(t)Ax(t) + 2\alpha \sum_{i=1}^{n} d_i \int_0^{x_i(t)} f_i(s)ds \\ & + \sum_{i=1}^{N} \frac{1}{1-\eta_i} \int_{t-\tau_i(t)}^{t} f^T(x(s))Q_i f(x(s))ds \\ & + \alpha \sum_{i=1}^{N} \int_{t-\tau_i(t)}^{t} f^T(x(s))P_i f(x(s))ds \end{aligned} \tag{83}$$

where $Q_i \geq 0, i = 1, \ldots, N$, and $\alpha > 0$ are to be defined later. The derivative of (83) along the trajectories of (19) is as follows:

$$\begin{aligned} & \dot{V}(x(t)) \\ & = 2x^T(t)A\left[-Ax(t)+Wf(x(t))+\sum_{i=1}^{N} W_i f(x(t-\tau_i(t)))\right] \\ & \quad + 2\alpha f^T(x(t))D\left[-Ax(t) + Wf(x(t)) + \sum_{i=1}^{N} W_i f(x(t-\tau_i(t)))\right] \\ & \quad + \sum_{i=1}^{N} \frac{1}{1-\eta_i}\left[f^T(x(t))Q_i f(x(t)) - (1-\dot{\tau}_i(t)) \times f^T(x(t-\tau_i(t)))Q_i f(x(t-\tau_i(t)))\right] \\ & \quad + \alpha \sum_{i=1}^{N}\left[f^T(x(t))P_i f(x(t)) - (1-\dot{\tau}_i(t)) \times f^T(x(t-\tau_i(t)))P_i f(x(t-\tau_i(t)))\right]. \end{aligned} \tag{84}$$

Because $\dot{\tau}_i(t) \leq \eta_i < 1$, it is clear that $(1 - \dot{\tau}_i(t)/1 - \eta_i) \geq 1$. Hence, from (84), we have

$$\begin{aligned} \dot{V}(x(t)) \leq {} & -2x^T(t)AAx(t) + 2x^T(t)AWf(x(t)) \\ & + 2x^T(t)A\sum_{i=1}^{N} W_i f(x(t-\tau_i(t))) \\ & - 2\alpha f^T(x(t))DAx(t) + 2\alpha f^T(x(t))DWf(x(t)) \\ & + 2\alpha f^T(x(t))D\sum_{i=1}^{N} W_i f(x(t-\tau_i(t))) \\ & + \sum_{i=1}^{N}\left[\frac{1}{1-\eta_i} f^T(x(t))Q_i f(x(t)) - f^T(x(t-\tau_i(t)))Q_i f(x(t-\tau_i(t)))\right] \\ & + \alpha\sum_{i=1}^{N}\left[f^T(x(t))P_i f(x(t)) - (1-\eta_i)f^T \times (x(t-\tau_i(t)))P_i f(x(t-\tau_i(t)))\right]. \end{aligned} \tag{85}$$

Because

$$-2x^T(t)AAx(t) = -x^T(t)AAx(t) - N\frac{1}{N}x^T(t)AAx(t)$$

then by Lemma 2.1 and Assumption 2.1, the following inequalities hold for $i = 1, \ldots, N$:

$$\begin{aligned} & -2\alpha f^T(x(t))DAx(t) \\ & \quad \leq -2\alpha f^T(x(t))DA\Delta^{-1}f(x(t)) \end{aligned} \tag{86}$$

$$\begin{aligned} & -x^T(t)AAx(t) + 2x^T(t)AWf(x(t)) \\ & \quad \leq f^T(x(t))W^TWf(x(t)) \end{aligned} \tag{87}$$

$$\begin{aligned} & -\frac{1}{N}x^T(t)AAx(t) + 2x^T(t)AW_i f(x(t-\tau_i(t))) \\ & \quad \leq Nf^T(x(t-\tau_i(t)))W_i^TW_i f(x(t-\tau_i(t))) \end{aligned} \tag{88}$$

$$\begin{aligned} & -\alpha(1-\eta_i)f^T(x(t-\tau_i(t)))P_i f(x(t-\tau_i(t))) \\ & + 2\alpha f^T(x(t))DW_i f(x(t-\tau_i(t))) \\ & \quad \leq \frac{\alpha}{1-\eta_i} f^T(x(t))DW_iP_i^{-1}W_i^TDf(x(t)). \end{aligned} \tag{89}$$

Let

$$Q_i = NW_i^TW_i. \tag{90}$$

Substituting (86)–(89) into (85) yields

$$\begin{aligned} & \dot{V}(x(t)) \\ & \leq f^T(x(t))\left(W^TW + \sum_{i=1}^{N}\frac{N}{1-\eta_i}W_i^TW_i\right)f(x(t)) \\ & \quad + \alpha f^T(x(t))\left[-2DA\Delta^{-1} + DW + W^TD + \sum_{i=1}^{N}\left(\frac{1}{1-\eta_i}DW_iP_i^{-1}W_i^TD + P_i\right)\right]f(x(t)) \\ & = f^T(x(t))\left(W^TW + \sum_{i=1}^{N}\frac{N}{1-\eta_i}W_i^TW_i\right)f(x(t)) \\ & \quad - \alpha f^T(x(t))(-\Omega)f(x(t)) \\ & \leq \left[\lambda_M\left(W^TW + \sum_{i=1}^{N}\frac{N}{1-\eta_i}W_i^TW_i\right) - \alpha\lambda_m(-\Omega)\right]\|f(x(t))\|^2. \end{aligned} \tag{91}$$

If we choose

$$\alpha > \frac{\lambda_M\left(W^TW + \sum_{i=1}^{N}\frac{N}{1-\eta_i}W_i^TW_i\right)}{\lambda_m(-\Omega)} > 0$$

then, from (91), we have $\dot{V}(x(t)) < 0$ for $\forall f(x(t)) \neq 0$. Note that $f(x(t)) \neq 0$ implies that $x(t) \neq 0$. Now, let $f(x(t)) = 0$ and $x(t) \neq 0$. In this case, $\dot{V}(x(t))$ is in the following form:

$$\begin{aligned}\dot{V}(x(t)) \leq &-2x^T(t)AAx(t) + 2x^T(t)A\sum_{i=1}^{N}W_if(x(t-\tau_i(t)))\\ &-\sum_{i=1}^{N}f^T(x(t-\tau_i(t)))Q_if(x(t-\tau_i(t)))\\ &-\alpha\sum_{i=1}^{N}(1-\dot{\tau}_i(t))f^T(x(t-\tau_i(t)))P_if(x(t-\tau_i(t))).\end{aligned} \tag{92}$$

Using (88) and (90), from (92), we obtain

$$\begin{aligned}\dot{V}(x(t)) &\leq -x^T(t)AAx(t)\\ &\quad -\alpha\sum_{i=1}^{N}(1-\dot{\tau}_i(t))f^T(x(t-\tau_i(t)))P_if(x(t-\tau_i(t)))\\ &\leq -x^T(t)AAx(t) < 0 \qquad \forall x(t) \neq 0.\end{aligned}$$

Now, we consider the case where $f(x(t)) = x(t) = 0$. In this case, $\dot{V}(x(t))$ takes the form

$$\begin{aligned}\dot{V}(x(t)) \leq &-\sum_{i=1}^{N}f^T(x(t-\tau_i(t)))Q_if(x(t-\tau_i(t)))\\ &-\alpha\sum_{i=1}^{N}(1-\dot{\tau}_i(t))f^T(x(t-\tau_i(t)))P_i\\ &\times f(x(t-\tau_i(t))).\end{aligned}$$

It is easily seen that $\dot{V}(x(t)) < 0$ for $\forall f(x(t-\tau_i(t))) \neq 0$.

Hence, we have proved that $\dot{V}(x(t)) = 0$ if and only if $f(x(t)) = x(t) = f(x(t-\tau_i(t))) = 0$. Otherwise, $\dot{V}(x(t)) < 0$. By Lyapunov stability theory, it follows that the origin of system (19) is globally asymptotically stable. This completes the proof of Theorem 3.3. ■

APPENDIX VI
PROOF OF THEOREM 3.5

Choose the following Lyapunov–Krasovskii functional:

$$\begin{aligned}V(x(t)) = &\, x^T(t)A^{-1}x(t)\\ &+\sum_{i=1}^{N}\int_{t-\tau_i}^{t}f^T(x(s))P_if(x(s))ds\\ &+\sum_{i=1}^{n}\frac{2}{a_i}\int_0^{x_i(t)}f_i(s)ds\end{aligned} \tag{93}$$

where $P_i = (N+1)W_i^TA^{-1}A^{-1}W_i + I$, $i = 1, \ldots, N$. The derivative of (93) along the trajectories of model (19) with constant delays is as follows:

$$\begin{aligned}\dot{V}(x(t)) = &-2x^T(t)x(t) + 2x^T(t)A^{-1}Wf(x(t))\\ &+2x^T(t)A^{-1}\sum_{i=1}^{N}W_if(x(t-\tau_i))\\ &+\sum_{i=1}^{N}\left[f^T(x(t))\left((N+1)W_i^TA^{-1}A^{-1}W_i\right.\right.\\ &\left.+I\right)f(x(t)) - f^T(x(t-\tau_i))\left((N+1)W_i^T\right.\\ &\left.\left.\times A^{-1}A^{-1}W_i + I\right)f(x(t-\tau_i))\right]\\ &-2f^T(x(t))x(t) + 2f^T(x(t))A^{-1}Wf(x(t))\\ &+2f^T(x(t))A^{-1}\sum_{i=1}^{N}W_if(x(t-\tau_i)).\end{aligned} \tag{94}$$

Because $-2x^T(t)x(t) = -(N+1)(1/N+1)x^T(t)x(t) - x^T(t)x(t)$, by Lemma 2.1 we have for $i = 1, \ldots, N$

$$\begin{aligned}&-\frac{1}{N+1}x^T(t)x(t) + 2x^T(t)A^{-1}Wf(x(t))\\ &\quad\leq (N+1)f^T(x(t))W^TA^{-1}A^{-1}Wf(x(t))\end{aligned} \tag{95}$$

$$\begin{aligned}&-\frac{1}{N+1}x^T(t)x(t) + 2x^T(t)A^{-1}W_if(x(t-\tau_i))\\ &\quad\leq (N+1)f^T(x(t-\tau_i))W_i^TA^{-1}A^{-1}W_if(x(t-\tau_i))\end{aligned} \tag{96}$$

$$\begin{aligned}&-f^T(x(t-\tau_i))f(x(t-\tau_i)) + 2f^T(x(t))A^{-1}W_if(x(t-\tau_i))\\ &\quad\leq f^T(x(t))A^{-1}W_iW_i^TA^{-1}f(x(t)).\end{aligned} \tag{97}$$

By Assumption 2.1, we have

$$-x^T(t)x(t) \leq -f^T(x(t))\Delta^{-2}f(x(t)) \tag{98}$$

$$-f^T(x(t))x(t) \leq -f^T(x(t))\Delta^{-1}f(x(t)). \tag{99}$$

Substituting (95)–(99) into (94) yields

$$\begin{aligned}\dot{V}(x(t)) \leq f^T(x(t))\Bigg[&-\Delta^{-2} - 2\Delta^{-1} + A^{-1}W + W^TA^{-1}\\ &+(N+1)W^TA^{-1}A^{-1}W\\ &+\sum_{i=1}^{N}\Big((N+1)W_i^TA^{-1}A^{-1}W_i + I\\ &+A^{-1}W_iW_i^TA^{-1}\Big)\Bigg]f(x(t)).\end{aligned}$$

Hence, if (39) holds, $\dot{V}(x(t)) < 0$ for all $f(x(t)) \neq 0$. Because $f(x(t)) \neq 0$ implies $x(t) \neq 0$, then when $f(x(t)) = 0$ and

$x(t) \neq 0, \dot{V}(x(t))$ satisfies

$$\begin{aligned}
&\dot{V}(x(t)) \\
&\quad = -2x^T(t)x(t) - \sum_{i=1}^{N} f^T(x(t-\tau_i))P_i f(x(t-\tau_i))) \\
&\qquad + 2x^T(t)A^{-1}\sum_{i=1}^{N} W_i f(x(t-\tau_i)) \\
&\quad = -x^T(t)x(t) - N\frac{1}{N}x^T(t)x(t) \\
&\qquad + 2x^T(t)A^{-1}\sum_{i=1}^{N} W_i f(x(t-\tau_i)) \\
&\qquad - \sum_{i=1}^{N} f^T(x(t-\tau_i))P_i f(x(t-\tau_i))) \\
&\quad \leq -x^T(t)x(t) \\
&\qquad + N\sum_{i=1}^{N} f^T(x(t-\tau_i))W_i^T A^{-1}A^{-1}W_i f(x(t-\tau_i))) \\
&\qquad - \sum_{i=1}^{N} f^T(x(t-\tau_i))\left[(N+1)W_i^T A^{-1}A^{-1}W_i + I\right] \\
&\qquad \times f(x(t-\tau_i))).
\end{aligned}$$

Therefore

$$\begin{aligned}
\dot{V}(x(t)) &\leq -x^T(t)x(t) - \sum_{i=1}^{N} f^T(x(t-\tau_i)) \\
&\quad \times \left[W_i^T A^{-1}A^{-1}W_i + I\right] f(x(t-\tau_i))) \\
&\leq -x^T(t)x(t) \\
&< 0
\end{aligned}$$

for $\forall x(t) \neq 0$. In addition, $V(x(t))$ is radially unbounded. Then, by Lyapunov stability theory, the origin of (19) is globally asymptotically stable if condition (39) holds. This completes the proof of the Theorem 3.5. ■

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