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CHAOTIFYING FUZZY HYPERBOLIC MODEL USING ADAPTIVE INVERSE OPTIMAL CONTROL APPROACH*

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In this paper, the problem of chaotifying the continuous-time fuzzy hyperbolic model (FHM) is studied. By tracking the dynamics of a chaotic system, a controller based on inverse optimal control and adaptive parameter tuning methods is designed to chaotify the FHM. Simulation results show that for any initial value the FHM can track a chaotic system asymptotically.

Keywords: Chaotification; fuzzy hyperbolic model; adaptive inverse optimal control.

1. Introduction

It is well known that most conventional control methods and many special techniques can be used for chaos control [Chen & Dong, 1998] whether the purpose is to reduce "bad" chaos or to introduce "good" chaos. Numerous control methodologies have been proposed, developed, tested and applied. Due to its great potential in nontraditional applications such as those found within the context of physical, chemical, mechanical, electrical, optical and particularly, biological and medical systems [Schiff *et al.*, 1994; Yang *et al.*, 1995], making a nonchaotic system chaos or strengthening the existing chaos, known as "chaotification" (also known as "anticontrol"), has attracted increasing attention in recent years. The process of chaos control is now understood as a transition from chaos to order and sometimes from order to chaos, depending on the purpose of the application in different circumstances.

Recent studies have shown that any discrete map can be chaotified in the sense of Devaney or Li–Yorke by a state-feedback controller with a uniformly bounded control-gain sequence designed to make all Lyapunov exponents of the controlled system strictly positive or arbitrarily assigned [Chen & Lai, 1997; Chen & Lai, 1998; Wang & Chen, 1999, 2000a, 2000b, 2000c]. Even if there are some research works showing that a class of continuous stable systems can be chaotified

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[Wang *et al.*, 2000; Wang *et al.*, 2001; Tang *et al.*, 2001; Sanchez *et al.*, 2001; Zhong *et al.*, 2001; Yang *et al.*, 2002; Lü *et al.*, 2002], the problem how to chaotify general nonlinear systems that cannot be linearized is still unsolved.

A systematic design procedure, the fuzzy control method, has been widely used to control chaotic systems. Great success has been achieved in various applications especially in situations where the dynamics of systems are so complex that it is impossible to construct an accurate model [Passino & Yurkovich, 1998; Tanaka *et al.*, 1998; Chen & Chen, 1999; Chen *et al.*, 1999]. In the present work we study the chaotification of a specific fuzzy model. First, such a study is of academic interest since a fuzzy model is usually a nonlinear system and its chaotification is a difficult task in general. Second, to obtain a chaotic fuzzy model, the common method is to use fuzzy modeling approach to model a chaotic system [Tanaka *et al.*, 1998; Chen *et al.*, 1999]. On the other hand, it is natural to ask whether a nonchaotic fuzzy system can be chaotified by control inputs. Part of the answer has been given in [Li *et al.*, 2002] where the continuous-time T-S fuzzy model was chaotified under certain conditions. Parallel to the T-S fuzzy model, a new continuous-time fuzzy model, called the fuzzy hyperbolic model (FHM), has recently been proposed [Zhang & Quan, 2001; Quan, 2001]. When modeling a complex plant, an FHM can be obtained without knowing much information about the real plant, and it is easy to design a controller with an FHM since the FHM satisfies the Lipschitz condition. In this paper, we focus on how to make an FHM chaotified. In fact, the FHM belong to a class of Lur'e systems. The problem of robust H_∞ synchronization of two Lur'e systems has been studied in [Suykens *et al.*, 1997; Suykens *et al.*, 1997; Suykens *et al.*, 1999]. In these studies, sufficient conditions for uniform synchronization with a bound on the synchronization error were derived in the form of matrix inequalities, and the problem of controller design is solved using a nonlinear optimization approach. The merits of these results are that the structures of the controllers are simple, and relatively large parameter mismatches are allowed such that the systems remain synchronized with a relatively small synchronization error bound. However, since the controllers are not adaptive, synchronization errors will not converge to zero. In our approach, for a controlled FHM to track the dynamics of a given continuous-time chaotic system asymptotically, we design a controller directly within the framework of inverse optimal control theory with adaptive parameter tuning methods. By using inverse optimal control theory, we guarantee that the controller is optimal with a meaningful cost functional, and by using adaptive parameter tuning methods, we make the tacking errors converge to zero asymptotically.

This paper is organized as follows. In Sec. 2, preliminaries about the FHM are reviewed. In Sec. 3, the problem considered in this paper is described. In Sec. 4, a controller that can chaotify the FHM is designed. Simulation results are presented in Sec. 5, and conclusions are given in Sec. 6.

2. Preliminaries

In this section we review some necessary preliminaries for the FHM.

Definition 1. Given a plant with n input variables $x = (x_1(t), \ldots, x_n(t))^T$ and n output variables $\dot{x} = (\dot{x}_1(t), \ldots, \dot{x}_n(t))^T$. If each output variable corresponds to a group of fuzzy rules which satisfies the following conditions:

(i) For each output variable $\dot{x}_l$, $l = 1, 2, \ldots, n$, the corresponding group of fuzzy rules has the following form:

$$\begin{aligned} R^j :\ & \text{IF } x_1 \text{ is } F_{x_1}, x_2 \text{ is } F_{x_2}, \ldots, \text{ and } x_{n_l} \text{ is } F_{x_{n_l}} \\ & \text{THEN } \dot{x}_l = c^{\pm}_{F_{x_1}} + c^{\pm}_{F_{x_2}} + \cdots + c^{\pm}_{F_{x_{n_l}}}, j = 1, \ldots, 2^{n_l}, \end{aligned}$$

where F_{x_i} $(i = 1, \ldots, n_l)$ are fuzzy sets of x_i, which include P_{x_i} (positive) and N_{x_i} (negative), and $c^{\pm}_{F_{x_{nl}}}$ $(i = 1, \ldots, n_l)$ are $2n_l$ real constants corresponding to F_{x_i};

(ii) The constant terms $c^{\pm}_{F_{x_i}}$ in the THEN-part correspond to F_{x_i} in the IF-part; that is, if the language value of F_{x_i} term in the IF-part is P_{x_i}, $c^{+}_{F_{x_i}}$ must appear in the THEN-part; if the language value of F_{x_i} term in the IF-part is N_{x_i}, $c^{-}_{F_{x_i}}$ must appear in the THEN-part; if there is no F_{x_i} in the IF-part, $c^{\pm}_{F_{x_i}}$ does not appear in the THEN-part;

(iii) There are 2^{n_l} fuzzy rules in each rule base; that is, there are a total of 2^{n_l} input variable combinations of all the possible P_{x_i} and N_{x_i} in the IF-part.

We call this group of fuzzy rules "hyperbolic type fuzzy rule base" (HFRB). To describe a plant with n output variables, we will need n HFRBs.

Theorem 1 [Quan, 2001; Zhang & Quan, 2001]. *Given n HFRBs, if we define the membership function of P_{x_i} and N_{x_i} as:*

$$\begin{aligned}\mu_{P_{x_i}}(x_i) &= e^{-\frac{1}{2}(x_i-k_i)^2}\\ \mu_{N_{x_i}}(x_i) &= e^{-\frac{1}{2}(x_i+k_i)^2}\end{aligned} \tag{1}$$

where $i = 1, \ldots, n$ and k_i are constants. Denoting $c^+_{F_{x_i}}$ by $c_{P_{x_i}}$ and $c^-_{F_{x_i}}$ by $c_{N_{x_i}}$, we can derive the following model:

$$\begin{aligned}\dot{x}_l &= f(x)\\ &= \sum_{i=1}^{n_l} \frac{c_{P_{x_i}} e^{k_i x_i} + c_{N_{x_i}} e^{-k_i x_i}}{e^{k_i x_i} + e^{-k_i x_i}}\\ &= \sum_{i=1}^{n_l} p_i + \sum_{i=1}^{n_l} q_i \frac{e^{k_i x_i} - e^{-k_i x_i}}{e^{k_i x_i} + e^{-k_i x_i}}\\ &= \sum_{i=1}^{n_l} p_i + \sum_{i=1}^{n_l} q_i \tanh(k_i x_i)\end{aligned} \tag{2}$$

where

$$p_i = \frac{c_{P_{x_i}} + c_{N_{x_i}}}{2} \quad \text{and} \quad q_i = \frac{c_{P_{x_i}} - c_{N_{x_i}}}{2}\,.$$

Therefore, the whole system has the following form:

$$\dot{x} = P + A\tanh(Kx) \tag{3}$$

where P is a constant vector, A is a constant matrix, and $\tanh(Kx)$ *is defined by*

$$\tanh(Kx) = [\tanh(k_1 x_1),\ \tanh(k_2 x_2), \ldots, \tanh(k_n x_n)]^T.$$

We will call (3) a fuzzy hyperbolic model (FHM).

Let Y be the space composed of all the functions having the form of the right-hand side of (2). We then have the following theorem.

Theorem 2. *For any given real continuous g on the compact set $U \subset R^n$ and arbitrary $\varepsilon > 0$, there exists an $f \in Y$ such that*

$$\sup_{x\in U} |g(x) - f(x)| < \varepsilon\,.$$

Proof. See Appendix A. ■

From Definition 1, if we set $c_{P_{x_i}}$ and $c_{N_{x_i}}$ negative to each other, we can obtain a homogeneous FHM:

$$\dot{x} = A\tanh(Kx)\,. \tag{4}$$

The homogeneous FHM given in (4) will be studied in this paper for chaotification through the use of an adaptive controller.

3. Problem Description

Since the difference between (3) and (4) is only the constant vector term in (3), there is essentially no difference between the control of (3) and (4). In this section, we will design a fuzzy controller for homogeneous FHM that can chaotify the model in (4).

Suppose that there exists a chaotic system having the following form:

$$\dot{x}_r = f_r(x_r),\ x_r \in R^n,\ f_r(\cdot) \in R^n\,. \tag{5}$$

Consider an uncontrolled FHM in the following form:

$$\dot{x} = Af(x) = A\tanh(Kx) \tag{6}$$

where x is the state and $A = [a_{ij}]_{n\times n}$ is a matrix. Due to the hyperbolic tangent form of $f(x)$, we know that $f^T(x)\cdot x \geq 0$ for all x, $f(x) = 0$ only at $x = 0$, and $\lim_{\|x\|_2\to\infty} f^T(x)\cdot x = +\infty$. Therefore, there exist positive constants γ_1 and γ_2 such that $\gamma_1\|x\|_2^2 \leq f^T(x)\cdot x \leq \gamma_2\|x\|_2^2$ [Miller & Michel, 1982].

The idea here is to design a controller u such that the controlled FHM

$$\dot{x} = Af(x) + u\,, \tag{7}$$

can track the dynamics of system (5), i.e.

$$\lim_{t\to\infty} \|e(t)\|_2 = \lim_{t\to\infty} \|x(t) - x_r(t)\|_2 = 0\,. \tag{8}$$

For each output $\dot{x}_l$, $l = 1, \ldots, n$, we choose to use the following controller fuzzy rule base (CFRB):

$$R_i:\ \text{IF}\ \ x_1 \text{ is } F_{x_1}, x_2 \text{ is } F_{x_2}, \ldots,\ \text{and } x_{n_l} \text{ is } F_{x_{n_l}}$$
$$\text{THEN}\ \ u_l = u_l^i(x)\ i = 1, \ldots, 2^{n_l}$$

That is to say, each CFRB has the same IF-part as that of the corresponding HFRB, and has each THEN-part given by a nonlinear function of the plant's state. By this design method, we get a controller as

$$u(x) = [u_1(x),\ u_2(x), \ldots, u_n(x)]^T\,, \tag{9}$$

where

$$u_l = \sum_{i=1}^{2^{n_l}} h_l^i(x) u_l^i(x),\ l = 1, \ldots, n \tag{10}$$

and

$$h_l^i(x) = \prod_{k=1}^{n_l} \mu^i_{F_{x_k}} \Big/ \sum_{i=1}^{2^{n_l}} \prod_{k=1}^{n_l} \mu^i_{F_{x_k}} . \tag{11}$$

In (11), $\mu^i_{F_{x_k}}$ is the membership function of F_{x_k} with the ith rule. It is easy to see that

$$\sum_{i=1}^{2^{n_l}} h_l^i(x) = 1 . \tag{12}$$

Adding this controller to (6), we obtain the controlled system (7).

Suppose that u has the form $u(x,t) = v(x) + w(x,t)$, where $v(x) = \Lambda x$ is the state-feedback, $\Lambda = -\mathrm{diag}[\lambda_i]_{n\times n} \in R^{n\times n}$ with λ_i $(i = 1, \ldots, n)$ real numbers, and $w(x,t)$ is to be designed. Then, (7) becomes

$$\dot{x} = \Lambda x + Af(x) + w(x,t) . \tag{13}$$

From (5) and (13), we obtain

$$\dot{e} = \Lambda x + Af(x) + w(x,t) - f_r(x_r) . \tag{14}$$

In practical control applications, since Λ is the state-feedback matrix and A is determined by the fuzzy rules, they may be affected by some uncertain factors such as parameter shifts and errors in modeling, and therefore, the parameters of system (14), Λ and A, would include some uncertainties. On the other hand, k_i $(i = 1, \ldots, n)$ can be fixed since they are determined by the membership functions chosen in modeling. Once the membership functions are fixed, k_i $(i, = 1, \ldots, n)$ are invariant. So in this paper, we assume Λ and A are tunable, and k_i $(i = 1, \ldots, n)$ are constants.

For system (13) to track the system (5), the following natural solvability assumption is needed [Li & Krstic, 1997].

Assumption 1. There exist functions $\rho(t)$ and $\alpha(t)$ such that

$$\begin{aligned} \frac{d\rho(t)}{dt} &= \Lambda_0 \rho(t) + A_0 f(\rho(t)) + \alpha(t) \\ \rho(t) &= x_r(t) \end{aligned} \tag{15}$$

where $A_0 = [a_{0ij}]_{n\times n}$ and $\Lambda_0 = -\mathrm{diag}[\lambda_{0i}]_{n\times n}$ are known constant matrices and λ_{0i} are positive real numbers for $i = 1, \ldots, n$.

From (5) and (15) the following equation can then be derived:

$$\Lambda_0 x_r + A_0 f(x_r) + \alpha(t) = f_r(x_r) . \tag{16}$$

Substituting (16) into (14), we have

$$\begin{aligned} \dot{e} &= \Lambda_0 e + A_0[f(e + x_r) - f(x_r)] \\ &\quad + [w - \alpha(t)] + \tilde{\Lambda} x + \tilde{A} f(x) \end{aligned} \tag{17}$$

where $\tilde{\Lambda} = \Lambda - \Lambda_0 = -\mathrm{diag}[\lambda_i - \lambda_{0i}]_{n\times n}$ and $\tilde{A} = A - A_0 = [a_{ij} - a_{0ij}]_{n\times n} = [\tilde{a}_{ij}]_{n\times n}$. Let $\phi(e, x_r) = f(e + x_r) - f(x_r)$, $\tilde{u} = w - \alpha(t)$. Then, (17) can be rewritten as

$$\dot{e} = \Lambda_0 e + A_0 \phi(e, x_r) + \tilde{\Lambda} x + \tilde{A} f(x) + \tilde{u} . \tag{18}$$

Remark 1. It is clear that $\phi(e, x_r) = 0$ if $e = 0$. Moreover, $f(e + x_r) = A \tanh(K(e + x_r))$ is monotonically increasing (or decreasing) for each component e_i of e. Since $e_i > 0$ (or $e_i < 0$) implies that $e_i + x_{ri} > x_{ri}$ (or $e_i + x_{ri} < x_{ri}$) for all x_{ri}, $f_i(e_i + x_{ri}) > f_i(x_{ri})$ (or $f_i(e_i + x_{ri}) < f_i(x_{ri})$). This means that $\phi^T(e, x_r)e = (f(e + x_r) - f(x_r))^T e > 0$. Therefore, there exist positive constants γ_1, γ_2, and L_ϕ such that

$$\gamma_1 \|e\|_2^2 \le \phi^T(e, x_r)e \le \gamma_2 \|e\|_2^2 \tag{19}$$

and

$$\|\phi(e, x_r)\|_2 < L_\phi \|e\|_2 . \tag{20}$$

Therefore, $\phi(e, x_r)$ is Lipschitz with respect to e.

4. Controller Design

We first state the following lemma that is required in our controller design.

Lemma 4.1 [Sanchez, 2001]. *For all matrices X, $Y \in R^{n\times k}$ and $Q \in R^{n\times n}$ with $Q = Q^T > 0$, the following inequality holds*:

$$X^T Y + Y^T X \le X^T Q X + Y^T Q^{-1} Y . \tag{21}$$

Theorem 3. *For system (14), if the controller is chosen as*

$$\begin{aligned} w &= -(A_0^T A_0 + I)\phi(e, x_r) - \Lambda_0 x_r \\ &\quad - A_0 f(x_r) + f_r(x_r) \end{aligned} \tag{22}$$

and the parameter adaptive update laws are chosen as

$$\begin{aligned}\dot{\lambda}_i &= -\phi_i(e, x_r)x_i\,,\\ \dot{a}_{ij} &= -\phi_i(e, x_r)f_j(x)\,,\end{aligned} \tag{23}$$

for $i = 1, \ldots, n$, $j = 1, \ldots, n$, *in which* $\phi_i(e, x_r)$ *and* x_i *are the* i*th component of* $\phi(e, x_r)$ *and* x, *respectively, and* $f_j(x)$ *is the* j*th component of* $f(x)$, *then the state of system (18),* e, *is globally asymptotically stable, i.e.*

$$\lim_{t\to\infty} \|e(t)\|_2 = 0\,.$$

Proof. Using (16), we can get $\alpha(t) = f_r(x_r) - \Lambda_0 x_r - A_0 f(x_r)$. Thus,

$$\begin{aligned}\tilde{u} &= w(x,t) - \alpha(t)\\ &= -(A_0^T A_0 + I)\phi(e, x_r)\,.\end{aligned} \tag{24}$$

Define

$$\begin{aligned}\varepsilon &\triangleq [e^T(t),\ \theta^T(t)]^T\\ &= [e^T(t),\ \tilde{\lambda}_1(t),\ \ldots, \tilde{\lambda}_n(t),\ \tilde{a}_{11}(t), \ldots,\\ &\qquad \tilde{a}_{1n}(t),\ \tilde{a}_{21}(t), \ldots, \tilde{a}_{nn}(t)]^T\,.\end{aligned}$$

We choose

$$\begin{aligned}V(\varepsilon) &= \sum_{i=1}^{n} \int_0^{e_i} \phi_i(\eta, x_r)d\eta_i\\ &\quad + \frac{1}{2}\sum_{i=1}^{n}\tilde{\lambda}_i^2 + \frac{1}{2}\sum_{i=1,j=1}^{n}\tilde{a}_{ij}^2\,,\end{aligned} \tag{25}$$

where η_i is the ith element of η.

Because of (19), we know that $V(\varepsilon)$ is radially unbounded, i.e. $V(\varepsilon) > 0$ for all ε and $V(\varepsilon) \to \infty$ as $\|\varepsilon\|_2 \to \infty$. Its time-derivative is as follows:

$$\begin{aligned}\dot{V}(\varepsilon) &= \phi^T(e, x_r)(\Lambda_0 e + A_0\phi(e, x_r)\\ &\quad + \tilde{\Lambda}x + \tilde{A}f(x) + \tilde{u})\\ &\quad + \sum_{i=1}^{n}\tilde{\lambda}_i\dot{\lambda}_i + \sum_{i=1,j=1}^{n}\tilde{a}_{ij}\dot{a}_{ij}\\ &= \phi^T(e, x_r)\Lambda_0 e + \phi^T(e, x_r)A_0\phi(e, x_r)\\ &\quad + \phi^T(e, x_r)(\tilde{\Lambda}x + \tilde{A}f(x))\\ &\quad + \phi^T(e, x_r)\tilde{u} + \sum_{i=1}^{n}\tilde{\lambda}_i\dot{\lambda}_i + \sum_{i=1,j=1}^{n}\tilde{a}_{ij}\dot{a}_{ij}\\ &\triangleq L_{\overline{f}}V + (L_g V)\tilde{u}\end{aligned} \tag{26}$$

where

$$\begin{aligned}L_{\overline{f}}V &\triangleq \phi^T(e, x_r)\Lambda_0 e + \phi^T(e, x_r)A_0\phi(e, x_r)\\ &\quad + \phi^T(e, x_r)(\tilde{\Lambda}x + \tilde{A}f(x))\\ &\quad + \sum_{i=1}^{n}\tilde{\lambda}_i\dot{\lambda}_i + \sum_{i=1,j=1}^{n}\tilde{a}_{ij}\dot{a}_{ij}\end{aligned} \tag{27}$$

and

$$L_g V \triangleq \phi^T(e, x_r)\,. \tag{28}$$

Applying Lemma 1 with $Q = I$, we get

$$\begin{aligned}\dot{V}(\varepsilon) &= \phi^T(e, x_r)\Lambda_0 e + \frac{1}{2}\phi^T(e, x_r)\phi(e, x_r)\\ &\quad + \frac{1}{2}\phi^T(e, x_r)A_0^T A_0\phi(e, x_r)\\ &\quad + \phi^T(e, x_r)\tilde{u} + \phi^T(e, x_r)(\tilde{\Lambda}(t)x\\ &\quad + \tilde{A}(t)f(x)) + \sum_{i=1}^{n}\tilde{\lambda}_i\dot{\lambda}_i + \sum_{i=1,j=1}^{n}\tilde{a}_{ij}\dot{a}_{ij}\\ &= \phi^T(e, x_r)\Lambda_0 e + \frac{1}{2}\phi^T(e, x_r)\phi(e, x_r)\\ &\quad + \frac{1}{2}\phi^T(e, x_r)A_0^T A_0\phi(e, x_r) + \phi^T(e, x_r)\tilde{u}\\ &\quad + \sum_{i=1}^{n}\tilde{\lambda}_i(\dot{\lambda}_i + \phi_i(e, x_r)x_i)\\ &\quad + \sum_{i=1,j=1}^{n}\tilde{a}_{ij}(\dot{a}_{ij} + \phi_i(e, x_r)f_j(x))\,.\end{aligned}$$

Substituting (23) into the equality above and using inequalities (19) and (20), we obtain

$$\begin{aligned}\dot{V}(\varepsilon) &\le -\left(\lambda^*\gamma_1 - \frac{1}{2}L_\phi^2\right)\|e\|_2^2\\ &\quad + \frac{1}{2}\phi^T(e, x_r)A_0^T A_0\phi(e, x_r)\\ &\quad + \phi^T(e, x_r)\tilde{u}\end{aligned} \tag{29}$$

where $\lambda^* = \min\{\lambda_{0i}; i = 1, \ldots, n\}$.

If we let $R^{-1}(\varepsilon) = (1/\beta)(A_0^T A_0 + I)$, where $\beta \ge 2$ is a constant, we have

$$\begin{aligned}-\beta R^{-1}(\varepsilon)(L_g V)^T &= \tilde{u}\\ &= -(A_0^T A_0 + I)\phi(e, x_r)\,.\end{aligned} \tag{30}$$

In general $R^{-1}(\varepsilon)$ is a function of ε, but for our purpose it is chosen as a constant matrix. The motivation for this operation will be seen from the inverse optimization problem to be discussed later.

Substituting (24) into (29), we get

$$\dot V(\varepsilon) \le -\left(\lambda^*\gamma_1 - \frac{1}{2}L_\phi^2\right)\|e\|_2^2 - \frac{1}{2}\|A_0^TA_0\|L_\phi^2\|e\|_2^2 - L_\phi^2\|e\|_2^2 = -\left(\lambda^*\gamma_1 + \frac{1}{2}\|A_0^TA_0\|L_\phi^2 + \frac{1}{2}L_\phi^2\right)\|e\|_2^2 \le 0\,. \tag{31}$$

By LaSalle's invariance principle [Slotine & Li, 1991], we know that the invariant set of (15), *IS*, has the following form:

$$IS = \{\varepsilon|\dot V = 0\} = \{\varepsilon|(\mathbf{0}, \theta^T)\}\,.$$

This completes the proof of the theorem. ■

Remark 2. From the proof of Theorem 3, we can find that even in the case $\lim_{t\to\infty}\|e(t)\|_2 = 0$, the limits $\lim_{t\to\infty}\|\tilde\Lambda(t)\|_2$ and $\lim_{t\to\infty}\|\tilde A(t)\|_2$ may not be zero. This conclusion can also be drawn from (23), from which we get $\phi(e, x_r) = 0$ if $e = 0$. This implies that $\Lambda(t)$ and $A(t)$ approach some constants which may be different from Λ_0 and A_0.

To avoid the burden that the partial differential equation of Hamillton–Jacobi–Bellman (HJB) imposes on the problem of optimal control of nonlinear systems, inverse optimal control theory has been developed recently. The difference between the traditional optimization and inverse optimal control problem is that, the former seeks a controller that minimizes a given cost, while the latter is concerned with finding a controller that minimizes some "meaningful" cost.

According to Li and Krstic [1997] and Krstic and Li [1998], for the inverse optimal control problem of system (18) to be solvable under the control of (23) and (24) we need to find a positive real-valued function $R(\varepsilon)$ and a positive definite function $l(\varepsilon)$ such that the following cost functional

$$J(\tilde u) = \lim_{t\to\infty}\{2\beta V(\varepsilon(\tau)) + \int_0^t (l(\varepsilon(\tau)) + \tilde u(\tau)^T R(\varepsilon(\tau))\tilde u(\tau))d\tau\}, \quad \beta \ge 2 \tag{32}$$

is minimized.

In the following, we will show that the controller we have designed can indeed solve the inverse optimal control problem.

Theorem 4. *If we choose*

$$l(\varepsilon) = -2\beta L_{\overline f}V + 2\beta(L_gV)R^{-1}(\varepsilon)(L_gV)^T + \beta(\beta-2)((L_gV)R^{-1}(\varepsilon)(L_gV)^T) = -2\beta L_{\overline f}V + \beta^2(L_gV)R^{-1}(\varepsilon)(L_gV)^T \tag{33}$$

and

$$R(\varepsilon) = \beta(A_0^TA_0 + I)^{-1}\,, \quad \beta \ge 2\,, \tag{34}$$

the cost functional (32) for system (18) under the parameter update laws (23) and the state feedback law (24) will be minimized.

Proof. To prove this theorem, first we should prove that $R(\varepsilon)$ is positive and symmetry, and $l(\varepsilon)$ is radially unbounded, i.e. $R(\varepsilon) = R^T(\varepsilon) > 0$ and $l(\varepsilon) > 0$ for all $\varepsilon \ne 0$ and $l(\varepsilon) \to +\infty$ as $\varepsilon \to \infty$. It is clear that $R(\varepsilon)$ chosen according to (34) satisfies this requirement. Using (23), (24), (27), (28), and (34), we obtain

$$l(\varepsilon) = 2\beta\phi^T(e, x_r)\Lambda_0 e - 2\beta\phi^T(e, x_r)A_0\phi(e, x_r) + \beta\phi^T(e, x_r)(A_0^TA_0 + I)\phi(e, x_r)\,. \tag{35}$$

Applying Lemma 1 to the second term on the right-hand side of (35), we get

$$l(\varepsilon) \ge 2\beta\lambda^*\phi^T(e, x_r)e - \beta\phi^T(e, x_r)\phi(e, x_r) - \beta\phi^T(e, x_r)A_0^TA_0\phi(e, x_r) + \beta\phi^T(e, x_r)(A_0^TA_0 + I)\phi(e, x_r) \ge 2\beta\lambda^*\phi^T(e, x_r)e\,. \tag{36}$$

This means that $l(\varepsilon)$ is radially unbounded. Substituting (30) into (26), we get

$$\dot V = L_{\overline f}V + (L_gV)(-\beta R^{-1}(\varepsilon))(L_gV)^T\,. \tag{37}$$

Multiplying it by -2β, we obtain

$$-2\beta\dot V(\varepsilon(t)) = -2\beta L_{\overline f}V + 2\beta^2(L_gV)R^{-1}(\varepsilon)(L_gV)^T\,. \tag{38}$$

Considering (30) and (33), we get

$$l(\varepsilon) + \tilde u^TR(\varepsilon)\tilde u = -2\beta\dot V(\varepsilon(t))\,. \tag{39}$$

Substituting (39) into (32), we have

$$J(\tilde u) = \lim_{t\to\infty}\left\{2\beta V(\varepsilon(t)) + \int_0^t -2\beta\dot V(\varepsilon(\tau))d\tau\right\} = \lim_{t\to\infty}\{2\beta V(\varepsilon(t)) - 2\beta V(\varepsilon(t)) + 2\beta V(\varepsilon(0))\} = 2\beta V(\varepsilon(0))\,. \tag{40}$$

Thus, the minimum of the cost functional is $J(\tilde{u}) = 2\beta V(\varepsilon(0))$ for the optimal control law (23) and (24). The theorem is proved. ■

We now summarize the results presented above in the next theorem.

Theorem 5. *If we choose feedback control law* $u = v + w$ *in which* $v = \Lambda x$ *is the linear state feedback with* Λ *a diagonal constant matrix and* $w = -(A_0^T A_0 + I)\phi(e, x_r) - \Lambda_0 x_r - A_0 f(x_r) + f_r(x_r)$ *is the nonlinear feedback, and at the same time we choose parameter update laws for* Λ *and* A *according to (23), the controlled FHM (7) will be chaotified through minimizing the cost functional (32).*

Remark 3. Here, we should note that the designed controller is not unique since we have much freedom to select Λ_0 and A_0. In fact, it is necessary to select proper Λ_0 and A_0 so that the controller's energy satisfies requirements in practical applications. Once Λ_0 and A_0 are fixed, the controller is optimal which minimizes some "meaningful" cost. Theorem 5 also indicates that we can use FHM as a general device to produce various chaotic dynamics.

In practice, the control signal $u_l(x)$ in (10) is defined by $u_l^i(x)$ $(i = 1, \ldots, 2^{n_l}; l = 1, \ldots, n)$. Solutions for u_l^i in (10) are not unique since there are 2^{n_l} unknown variables to be solved from only one equation. In our study, we use a heuristic approach. That is, suppose for each $l = 1, \ldots, n$, $u_l^i(x) = \overline{u}_l^i \in [a_l^i, b_l^i]$ $(i = 1, \ldots, 2^{n_l} - 1)$, where $\overline{u}_l^i$ are real numbers and $[a_l^i, b_l^i]$ are closed intervals specified in advance. The parameters a_l^i and b_l^i can be fixed by taking into account the requirements of control signals. A special case of such solutions can be obtained when each $u_l^i(x)$ is equal to a nonlinear function $\overline{u}_l(x)$. In this case, the design of a fuzzy controller becomes equivalent to the design of a normal controller.

5. Simulation Results

Suppose that we have the following HFRBs:

If x_1 is P_{x_1} and x_2 is P_{x_2}, then $\dot{x}_3 = C_{x_1} + C_{x_2}$;
If x_1 is N_{x_1} and x_2 is P_{x_2}, then $\dot{x}_3 = -C_{x_1} + C_{x_2}$;
If x_1 is P_{x_1} and x_2 is N_{x_2}, then $\dot{x}_3 = C_{x_1} - C_{x_2}$;
If x_1 is N_{x_1} and x_2 is N_{x_2}, then $\dot{x}_3 = -C_{x_1} - C_{x_2}$;

If x_1 is P_{x_1} and x_3 is P_{x_3}, then $\dot{x}_2 = C_{x_1} + C_{x_3}$;
If x_1 is N_{x_1} and x_3 is P_{x_3}, then $\dot{x}_2 = -C_{x_1} + C_{x_3}$;
If x_1 is P_{x_1} and x_3 is N_{x_3}, then $\dot{x}_2 = C_{x_1} - C_{x_3}$;
If x_1 is N_{x_1} and x_3 is N_{x_3}, then $\dot{x}_2 = -C_{x_1} - C_{x_3}$;

If x_2 is P_{x_2} and x_3 is P_{x_3}, then $\dot{x}_1 = C_{x_2} + C_{x_3}$;
If x_2 is N_{x_2} and x_3 is P_{x_3}, then $\dot{x}_1 = -C_{x_2} + C_{x_3}$;
If x_2 is P_{x_2} and x_3 is N_{x_3}, then $\dot{x}_1 = C_{x_2} - C_{x_3}$;
If x_2 is N_{x_2} and x_3 is N_{x_3}, then $\dot{x}_1 = -C_{x_2} - C_{x_3}$.

Here, we choose membership functions of P_{x_i} and N_{x_i} as follows:

$$\begin{aligned} \mu_{P_{x_i}}(x) &= e^{-\frac{1}{2}(x_i - k_i)^2}, \\ \mu_{N_{x_i}}(x) &= e^{-\frac{1}{2}(x_i + k_i)^2}. \end{aligned} \tag{41}$$

Then, we have the following three-dimensional model:

$$\dot{x} = Af(x) = A\tanh(Kx) \tag{42}$$

where $x = [x_1, x_2, x_3]^T$,

$$A = \begin{bmatrix} 0 & C_{x_2} & C_{x_3} \\ C_{x_1} & 0 & C_{x_3} \\ C_{x_1} & C_{x_2} & 0 \end{bmatrix}$$

and $\tanh(Kx) = [\tanh(k_1 x_1), \tanh(k_2 x_2), \tanh(k_3 x_3)]^T$.

For (42), we choose the following CFRBs:

If x_1 is P_{x_1} and x_2 is P_{x_2}, then $u_3 = u_3^1(x)$;
If x_1 is N_{x_1} and x_2 is P_{x_2}, then $u_3 = u_3^2(x)$;
If x_1 is P_{x_1} and x_2 is N_{x_2}, then $u_3 = u_3^3(x)$;
If x_1 is N_{x_1} and x_2 is N_{x_2}, then $u_3 = u_3^4(x)$;

If x_1 is P_{x_1} and x_3 is P_{x_3}, then $u_2 = u_2^1(x)$;
If x_1 is N_{x_1} and x_3 is P_{x_3}, then $u_2 = u_2^2(x)$;
If x_1 is P_{x_1} and x_3 is N_{x_3}, then $u_2 = u_2^3(x)$;
If x_1 is N_{x_1} and x_3 is N_{x_3}, then $u_2 = u_2^4(x)$;

If x_2 is P_{x_2} and x_3 is P_{x_3}, then $u_1 = u_1^1(x)$;
If x_2 is N_{x_2} and x_3 is P_{x_3}, then $u_1 = u_1^2(x)$;
If x_2 is P_{x_2} and x_3 is N_{x_3}, then $u_1 = u_1^3(x)$;
If x_2 is N_{x_2} and x_3 is N_{x_3}, then $u_1 = u_1^4(x)$;

Then the controlled system is

$$\dot{x} = Af(x) + u = \Lambda x + Af(x) + w, \tag{43}$$

where $\Lambda = \mathrm{diag}[\lambda_1, \lambda_2, \lambda_3]$. Here, because of the special form of A, only three adaptive update laws will be required.

Suppose that the chaotic system we want to track is the Lorenz system:

$$\dot{x}_r = f_r(x_r) \tag{44}$$

where $x_r = [x_{1r}, x_{2r}, x_{3r}]^T$ and $f_r(x_r) = [a(x_{2r} - x_{1r}), cx_{1r} - x_{1r}x_{3r} - x_{2r}, x_{1r}x_{2r} - bx_{3r}]^T$. When

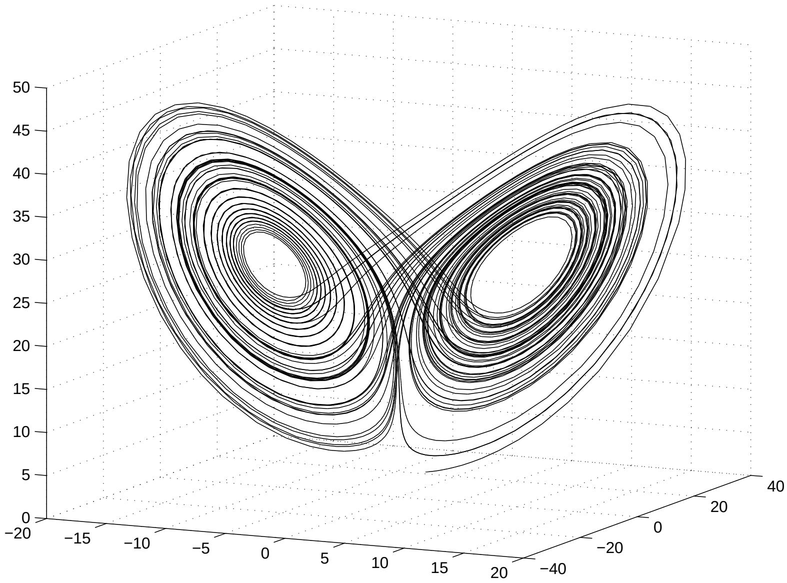

Fig. 1. Lorenz's chaotic attractor.

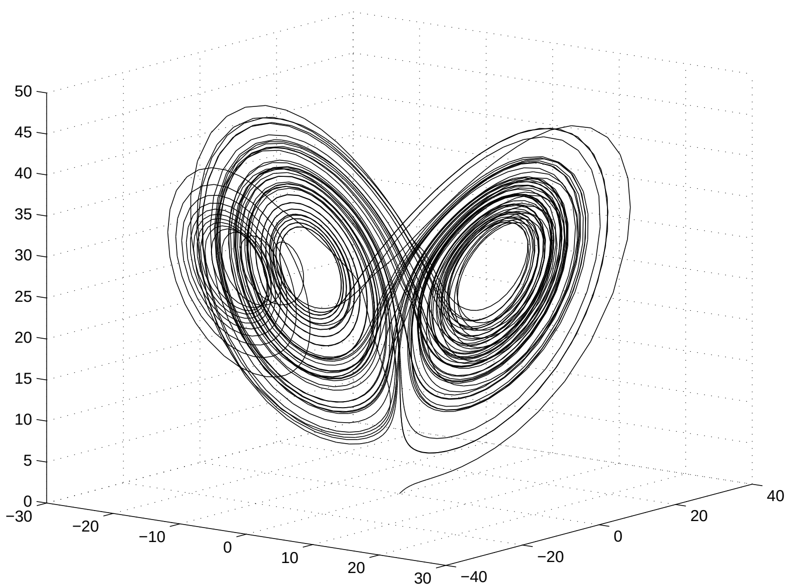

Fig. 2. The phase diagram of (43) when the tracking object is (44).

$a = 10$, $b = 8/3$, $c = 28$, the Lorenz system has a chaotic attractor shown in Fig. 1.

In this example, we choose

$$\Lambda_0 = \mathrm{diag}[-2,\ -2,\ -2]^T$$
$$\Lambda(0) = \mathrm{diag}[-1,\ -1,\ -2]^T,$$

$$A_0 = \begin{bmatrix} 0 & 3 & 4 \\ 3 & 0 & 4 \\ 3 & 3 & 0 \end{bmatrix}, \quad A(0) = \begin{bmatrix} 0 & 2.8 & 3.7 \\ 2.8 & 0 & 3.7 \\ 2.8 & 2.8 & 0 \end{bmatrix},$$

$[k_1,\ k_2,\ k_3] = [2,\ 3,\ 1]^T$, $x_r(0) = [2,\ 1,\ 3]^T$ and $x(0) = [0,\ 0,\ 0]^T$. We choose these matrices in our simulation according to the following guidelines. (1) Λ_0 is a diagonal matrix with negative diagonal elements; (2) $\Lambda(0)$ is a perturbation of Λ_0; (3) From the process of fuzzy modeling, we know that each element of matrix A_0 is either positive or zero; (4) $A(0)$ is a perturbation of A_0.

The simulation results are shown in Figs. 2–4. From these figures, we can see that the controlled

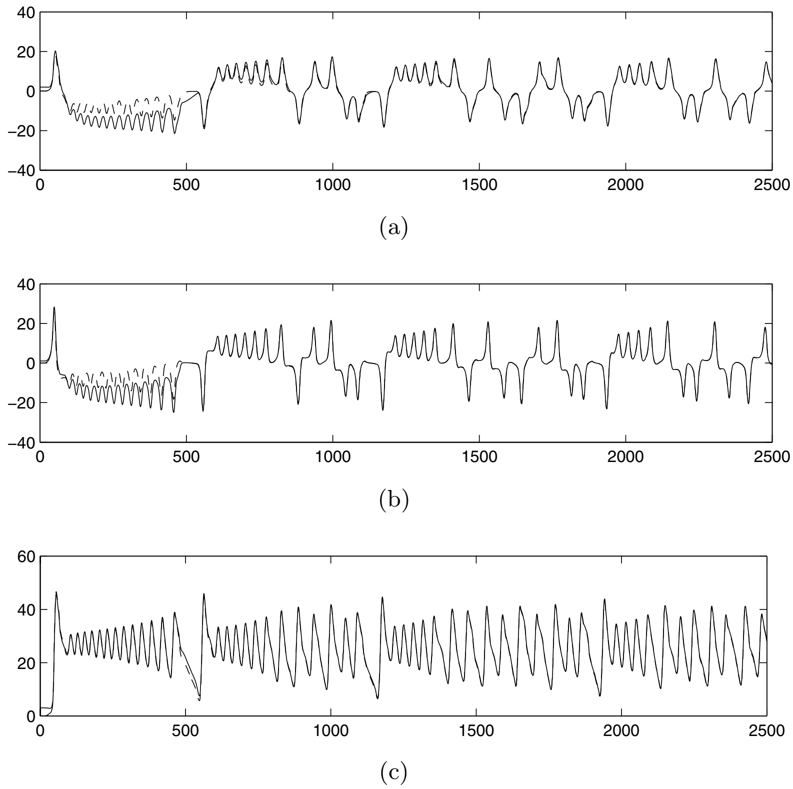

Fig. 3. State curves of (43) when tracking object is (44). The solid lines are the curves of (43) and the dashed lines are the curves of (44).

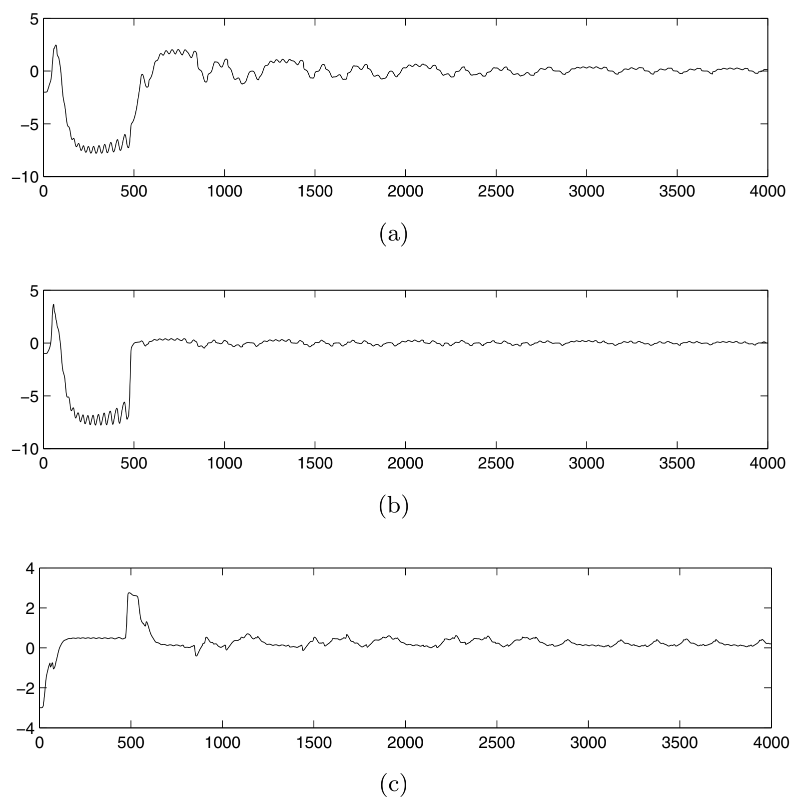

Fig. 4. The curve of (a) $x_1 - x_{r1}$; (b) $x_2 - x_{r2}$; (c) $x_3 - x_{r3}$.

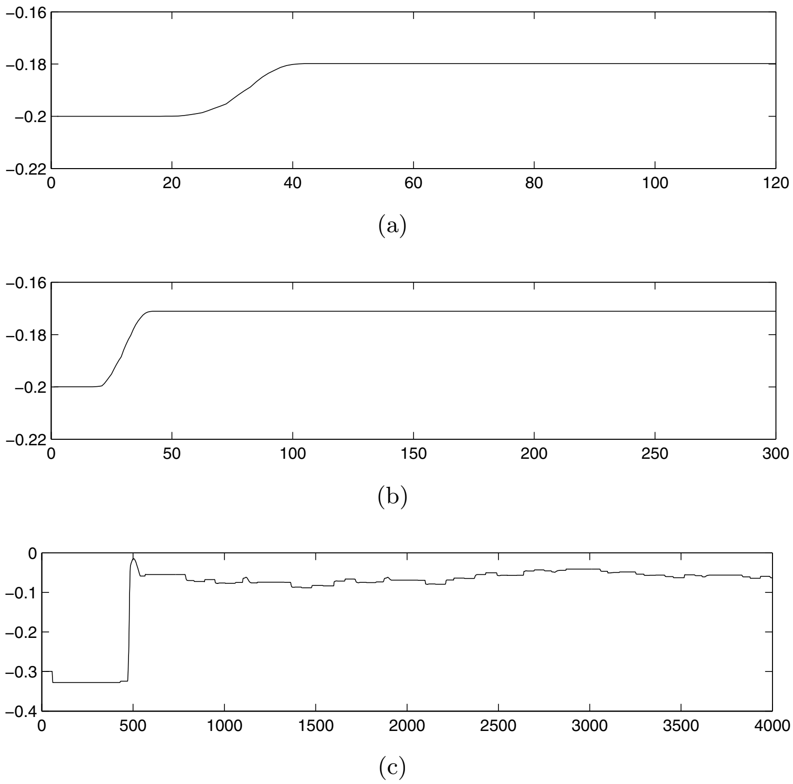

Fig. 5. The curve of (a) $C_{x_1} - C_1^0$; (b) $C_{x_2} - C_2^0$; (c) $C_{x_3} - C_3^0$.

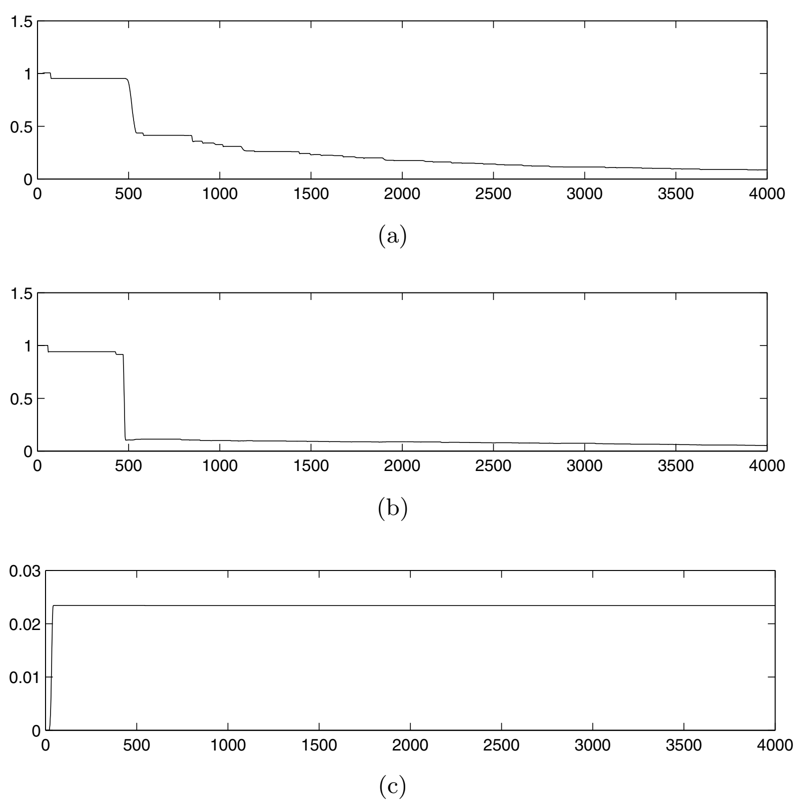

Fig. 6. The curve of (a) $\lambda_1 - \lambda_{10}$; (b) $\lambda_2 - \lambda_{20}$; (c) $\lambda_3 - \lambda_{30}$.

system (43) produces chaotic dynamics that have the same topological structure as (44) and the two systems' states become indistinguishable after a short period of time. Figures 5 and 6 show that the parameters also approach some constants, which is in accord with Remark 2. To get practical control $u_l^i(x)$ ($l = 1, 2, 3$ and $i = 1, 2, 3, 4$) in this simulation, we choose $u_l^1(x) = u_l^2(x) = u_l^3(x) = u_l^4(x)$ ($l = 1, 2, 3$).

6. Conclusions

In this paper, the chaotification of a fuzzy model by the use of an adaptive controller is studied. The design method for the controller is under the framework of inverse optimal control and parameter adaptation. The controller is designed for chaotifying the fuzzy hyperbolic model. The effectiveness of our design is shown through simulation. In our simulation studies, we have shown that the present design can track the Lorenz chaotic system. We believe that our design is novel in terms of the use of inverse optimal control and parameter adaptation for chaotifying the fuzzy hyperbolic model.

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Appendix A

The proof of Theorem 2 was originally given in [Quan, 2001]. We state it here for the purpose of completeness. To prove Theorem 2, we need the following lemma.

Lemma A.1 (Stone–Weierstrass Theorem [Rudin, 1964]). *Let Z be a set of real continuous functions on a compact set U. If (1) Z is an algebra, i.e. the set Z is closed under addition, multiplication, and scalar multiplication; (2) Z separates points on U, i.e. for every $x, y \in U, x \neq y$, there exists $f \in Z$ such that $f(x) \neq f(y)$; and (3) Z vanishes at no points of U, i.e. for each $x \in U$ there exists $f \in Z$ such that $f(x) \neq 0$; then the uniform closure of Z consists of all real continuous functions on U; i.e. (Z, d_∞), is dense in $(C[U], d_\infty)$, here $d_\infty(f_1, f_2) = \sup_{x\in U} |f_1(x) - f_2(x)|$.*

Using this lemma, we can prove Theorem 2.

Proof of Theorem 2. First, we prove that (Y, d_∞) is an algebra. For simplicity, we denote $c_{P_{x_i}}$ as c_{P_i} and denote $c_{N_{x_i}}$ as c_{N_i}. Let $f_1, f_2 \in Y$. We can write them as [cf. (2)]

$$f_1(x) = \sum_{i_1=1}^{m_1} \frac{c^1_{P_{i_1}} e^{k^1_{i_1} x_{i_1}} + c^1_{N_{i_1}} e^{-k^1_{i_1} x_{i_1}}}{e^{k^1_{i_1} x_{i_1}} + e^{-k^1_{i_1} x_{i_1}}} \tag{A.1}$$

$$f_2(x) = \sum_{i_2=1}^{m_2} \frac{c^2_{P_{i2}} e^{k^2_{i_2} x_{i_2}} + c^2_{N_{i_2}} e^{-k^2_{i_2} x_{i_2}}}{e^{k^2_{i_2} x_{i_2}} + e^{-k^2_{i_2} x_{i_2}}}. \tag{A.2}$$

We have

$$\begin{aligned} f_1(x) + f_2(x) &= \sum_{i_1=1}^{m_1} \frac{c^1_{P_{i_1}} e^{k^1_{i_1} x_{i_1}} + c^1_{N_{i_1}} e^{-k^1_{i_1} x_{i_1}}}{e^{k^1_{i_1} x_{i_1}} + e^{-k^1_{i_1} x_{i_1}}} \\ &\quad + \sum_{i_2=1}^{m_2} \frac{c^2_{P_{i2}} e^{k^2_{i_2} x_{i_2}} + c^2_{N_{i_2}} e^{-k^2_{i_2} x_{i_2}}}{e^{k^2_{i_2} x_{i_2}} + e^{-k^2_{i_2} x_{i_2}}} \\ &= \sum_{z=1}^{m_1+m_2} \frac{c_{P_z} e^{k_z x_z} + c_{N_z} e^{-k_z x_z}}{e^{k_z x_z} + e^{-k_z x_z}}. \end{aligned} \tag{A.3}$$

It is easy to see that (A.3) has the same form as (2); that is, $f_1 + f_2 \in Y$.

In the same way we can get

$$\begin{aligned} f_1(x) \cdot f_2(x) &= \sum_{i_1=1}^{m_1} \frac{c^1_{P_{i_1}} e^{k^1_{i_1} x_{i_1}} + c^1_{N_{i_1}} e^{-k^1_{i_1} x_{i_1}}}{e^{k^1_{i_1} x_{i_1}} + e^{-k^1_{i_1} x_{i_1}}} \times \sum_{i_2=1}^{m_2} \frac{c^2_{P_{i2}} e^{k^2_{i_2} x_{i_2}} + c^2_{N_{i_2}} e^{-k^2_{i_2} x_{i_2}}}{e^{k^2_{i_2} x_{i_2}} + e^{-k^2_{i_2} x_{i_2}}} \\ &= \sum_{i_1,i_2=1}^{m_1,m_2} \left(\frac{c^1_{P_{i_1}} e^{k^1_{i_1} x_{i_1}} + c^1_{N_{i_1}} e^{-k^1_{i_1} x_{i_1}}}{e^{k^1_{i_1} x_{i_1}} + e^{-k^1_{i_1} x_{i_1}}} \times \frac{c^2_{P_{i2}} e^{k^2_{i_2} x_{i_2}} + c^2_{N_{i_2}} e^{-k^2_{i_2} x_{i_2}}}{e^{k^2_{i_2} x_{i_2}} + e^{-k^2_{i_2} x_{i_2}}} \right) \\ &= \sum_{i_1,i_2=1}^{m_1,m_2} \left(\frac{Q}{(e^{k^1_{i_1} x_{i_1}} + e^{-k^1_{i_1} x_{i_1}})(e^{k^2_{i_2} x_{i_2}} + e^{-k^2_{i_2} x_{i_2}})} \right) \\ &= \sum_{i_1,i_2=1}^{m_1,m_2} \left(\frac{(c^{1*}_{P_{i_1}} + c^{2*}_{P_{i2}}) e^{k^1_{i_1} x_{i_1}} e^{k^2_{i_2} x_{i_2}} + (c^{1*}_{P_{i_1}} + c^{2*}_{N_{i_2}}) e^{k^1_{i_1} x_{i_1}} e^{-k^2_{i_2} x_{i_2}}}{(e^{k^1_{i_1} x_{i_1}} + e^{-k^1_{i_1} x_{i_1}})(e^{k^2_{i_2} x_{i_2}} + e^{-k^2_{i_2} x_{i_2}})} \right) \\ &\quad + \sum_{i_1,i_2=1}^{m_1,m_2} \left(\frac{(c^{1*}_{N_{i_1}} + c^{2*}_{P_{i2}}) e^{-k^1_{i_1} x_{i_1}} e^{k^2_{i_2} x_{i_2}} + (c^{1*}_{N_{i_1}} + c^{2*}_{N_{i_2}}) e^{-k^1_{i_1} x_{i_1}} e^{-k^1_{i_1} x_{i_1}}}{(e^{k^1_{i_1} x_{i_1}} + e^{-k^1_{i_1} x_{i_1}})(e^{k^2_{i_2} x_{i_2}} + e^{-k^2_{i_2} x_{i_2}})} \right) \end{aligned} \tag{A.4}$$

$$= \sum_{i_1,i_2=1}^{m_1,m_2} \left(\frac{c_{P_{i_1}}^{1*} e^{k_{i_1}^1 x_{i_1}} + c_{N_{i_1}}^{1*} e^{-k_{i_1}^1 x_{i_1}}}{e^{k_{i_1}^1 x_{i_1}} + e^{-k_{i_1}^1 x_{i_1}}} + \frac{c_{Pi2}^{2*} e^{k_{i_2}^2 x_{i_2}} + c_{N_{i_2}}^{2*} e^{-k_{i_2}^2 x_{i_2}}}{e^{k_{i_2}^2 x_{i_2}} + e^{-k_{i_2}^2 x_{i_2}}} \right)$$

$$= \sum_{z=1}^{m_1+m_2} \frac{c_{P_z} e^{k_z x_z} + c_{N_z} e^{-k_z x_z}}{e^{k_z x_z} + e^{-k_z x_z}}$$

where

$$\begin{aligned} Q = {} & c_{P_{i_1}}^1 c_{Pi2}^2 e^{k_{i_1}^1 x_{i_1}} e^{k_{i_2}^2 x_{i_2}} \\ & + c_{P_{i_1}}^1 c_{N_{i_2}}^2 e^{k_{i_1}^1 x_{i_1}} e^{-k_{i_2}^2 x_{i_2}} \\ & + c_{N_{i_1}}^1 c_{Pi2}^2 e^{-k_{i_1}^1 x_{i_1}} e^{k_{i_2}^2 x_{i_2}} \\ & + c_{N_{i_1}}^1 c_{N_{i_2}}^2 e^{-k_{i_1}^1 x_{i_1}} e^{-k_{i_1}^1 x_{i_1}} , \end{aligned}$$

and $c_{P_{i_1}}^{1*}$, $c_{N_{i_1}}^{1*}$, c_{Pi2}^{2*}, N_{i2}^{2*} satisfy the following equations:

$$\begin{aligned} c_{P_{i_1}}^{1*} + c_{P_{i_2}}^{2*} &= c_{P_{i_1}}^1 c_{P_{i_2}}^2 , \\ c_{P_{i_1}}^{1*} + c_{N_{i_2}}^{2*} &= c_{P_{i_1}}^1 c_{N_{i_2}}^2 \\ c_{N_{i_1}}^{1*} + c_{P_{i_2}}^{2*} &= c_{N_{i_1}}^1 c_{P_{i_2}}^2 , \\ c_{N_{i_1}}^{1*} + c_{N_{i_2}}^{2*} &= c_{N_{i_1}}^1 c_{N_{i_2}}^2 . \end{aligned} \tag{A.5}$$

It is easy to see that (A.4) is also in the same form as (2). Hence, $f_1 \cdot f_2 \in Y$.

Finally, for any constant $c \in R$, we have

$$\begin{aligned} cf(x) &= c \sum_{i=1}^{m} \frac{c_{P_i} e^{k_i x_i} + c_{N_i} e^{-k_i x_i}}{e^{k_i x_i} + e^{-k_i x_i}} \\ &= \sum_{i=1}^{m} \frac{c_{P_i}^* e^{k_i x_i} + c_{N_i}^* e^{-k_i x_i}}{e^{k_i x_i} + e^{-k_i x_i}} \end{aligned} \tag{A.6}$$

which is again in the same form as (2). Hence, $cf_1 \in Y$. Therefore, (Y, d_∞) is an algebra.

Next, we prove that (Y, d_∞) separates points on U. We prove this by constructing a required f; i.e. we specify $f \in Y$ such that $f(x^0) \neq f(y^0)$ for arbitrarily given x^0, $y^0 \in U$ with $x^0 \neq y^0$. Let $x^0 = (x_1^0, x_2^0, \ldots, x_n^0)^T$, $y^0 = (y_1^0, y_2^0, \ldots, y_n^0)^T$. If $x_i^0 \neq y_i^0$, choose input variable as

$$x_i^* = x_i - \frac{x_i^0 + y_i^0}{2} \tag{A.7}$$

$$k_i^* = \frac{x_i^0 - y_i^0}{2} . \tag{A.8}$$

That is, $x_i^* - k_i^* = x_i - x_i^0$ and $x_i^* + k_i^* = x_i - y_i^0$. Then, from (2) we can get

$$\begin{aligned} f(x^0) &= \sum_{i=1}^{n} \frac{c_{P_i} e^{-\frac{1}{2}(x_i^0 - x_i^0)^2} + c_{N_i} e^{-\frac{1}{2}(x_i^0 - y_i^0)^2}}{e^{-\frac{1}{2}(x_i^0 - x_i^0)^2} + e^{-\frac{1}{2}(x_i^0 - y_i^0)^2}} \\ &= \sum_{i=1}^{n} \frac{c_{P_i} + c_{N_i} e^{-\frac{1}{2}(x_i^0 - y_i^0)^2}}{1 + e^{-\frac{1}{2}(x_i^0 - y_i^0)^2}} \end{aligned} \tag{A.9}$$

$$\begin{aligned} f(y^0) &= \sum_{i=1}^{n} \frac{c_{P_i} e^{-\frac{1}{2}(y_i^0 - x_i^0)^2} + c_{N_i} e^{-\frac{1}{2}(y_i^0 - y_i^0)^2}}{e^{-\frac{1}{2}(y_i^0 - x_i^0)^2} + e^{-\frac{1}{2}(y_i^0 - y_i^0)^2}} \\ &= \sum_{i=1}^{n} \frac{c_{P_i} e^{-\frac{1}{2}(y_i^0 - x_i^0)^2} + c_{N_i}}{1 + e^{-\frac{1}{2}(y_i^0 - x_i^0)^2}} . \end{aligned} \tag{A.10}$$

Let $C_{P_i} = 1$ and $C_{N_i} = 0$. We have

$$\begin{aligned} f(x^0) - f(y^0) &= \sum_{i=1}^{n} \frac{1}{1 + e^{-\frac{1}{2}(x_i^0 - y_i^0)^2}} \\ &\quad - \sum_{i=1}^{n} \frac{e^{-\frac{1}{2}(x_i^0 - y_i^0)^2}}{1 + e^{-\frac{1}{2}(x_i^0 - y_i^0)^2}} \\ &= \frac{1 - \prod_{i=1}^{n} e^{-\frac{1}{2}(x_i^0 - y_i^0)^2}}{1 + \prod_{i=1}^{n} e^{-\frac{1}{2}(x_i^0 - y_i^0)^2}} . \end{aligned} \tag{A.11}$$

Since $x^0 \neq y^0$, there must be some i such that $x_i^0 \neq y_i^0$. Hence, we have $\prod_{i=1}^{n} e^{-\frac{1}{2}(x_i^0 - y_i^0)^2} \neq 1$. Thus, $f(x^0) \neq f(y^0)$. Therefore, (Y, d_∞) separates points on U. ■

Finally, we prove that (Y, d_∞) vanishes at no points of U. By observing (1) and (2), we simply choose all $c_{P_i} > 0$, $c_{N_i} > 0$ $(i = 1, 2, \ldots, m)$; that is, any $f \in Y$ with $c_{P_i} > 0$ and $c_{N_i} > 0$ serves as the required f.

From (2), it is obvious that Y is a set of real continuous functions on U. The universal approximation theorem is therefore a direct consequence of the Stone–Weierstrass Theorem.