Robust Exponential Stability of Recurrent Neural Networks With Multiple Time-Varying Delays

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***Abstract*—New criteria for the uniqueness and global robust exponential stability are established for the equilibrium point of interval recurrent neural networks with multiple time-varying delays via a decomposition method and analysis technique. Results are presented in the form of linear matrix inequality, which can be solved efficiently. Two numerical examples are employed to show the effectiveness of the present results.**

***Index Terms*—Multiple time-varying delays, recurrent neural networks, robust exponential stability.**

I. Introduction

SINCE THE linear matrix inequality (LMI) technique has been extensively applied to tackle various stability problems of neural networks and stabilization problems of control systems recently [3], [4], [7], [9], [11], [12], [16], [17], [19]–[21], [23], robust stability criteria in the form of LMI are proposed for interval neural networks with delays. To the best of the authors' knowledge, no robust stability criterion in the form of LMI has been reported for the recurrent neural network models studied in [5], [13]–[15]. The advantages of the stability results based on LMI include that not only they are easily verified using the interior-point algorithm, but also they consider the neuron's inhibitory and excitatory effects on neural networks, which overcomes the shortcomings in the results of [5], [14]. In many applications, to increase the convergence rate of neural network tending towards equilibrium point, it is very important to achieve global exponential stability [8], [13], [22]. Therefore, the global robust exponential stability analysis for recurrent neural networks studied in [5], [13]–[15] is significant from both theoretical and practical points of view.

In this paper, using a new decomposition method and analysis technique, we present some new LMI-based conditions for the uniqueness and global robust exponential stability for the equilibrium point of interval neural networks with multiple time-varying delays, which are computationally efficient.

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II. Problem Description and Preliminaries

Consider the following interval neural network model with multiple time-varying delays:

$$\dot{u}_i(t) = -a_i u_i(t) + \sum_{j=1}^{n} w_{ij}^0 g_j(u_j(t)) + \sum_{j=1}^{n} w_{ij} g_j(u_j(t-\tau_{ij}(t))) + U_i \quad (1)$$

where $u_i(t)$ is the ith neuron state, $w_{ij}^0 \in [\underline{w}_{ij}^0, \overline{w}_{ij}^0]$ and $w_{ij} \in [\underline{w}_{ij}, \overline{w}_{ij}]$ are constant connection coefficients, $a_i \in [\underline{a}_i, \overline{a}_i]$ with $\underline{a}_i > 0$ a positive constant, $g_j(u_j(t))$ is an activation function, U_i is an external input, $\tau_{ij}(t) > 0$ denotes the transmission delay, $i, j = 1, 2, \ldots, n$.

For convenience, let $W_0 = (w_{ij}^0)_{n\times n}, W_k = (w_{ij}^k)_{n\times n}$, whose kth row is composed of the kth row of matrix $W = (w_{ij})_{n\times n}$ and all other rows are zero, $k = 1, \ldots, n$. Let $A = \mathrm{diag}(a_1, \ldots, a_n), \tau_i(t) = (\tau_{i1}(t), \ldots, \tau_{in}(t))^T, u(t) = (u_1(t), \ldots, u_n(t))^T, g(u(t)) = (g_1(u_1(t)), \ldots, g_n(u_n(t)))^T, U = (U_1, \ldots, U_n)^T, g(u(t-\tau_i(t))) = (g_1(u_1(t-\tau_{i1}(t))), \ldots, g_n(u_n(t-\tau_{in}(t))))^T$, $i, j = 1, \ldots, n$. Then, model (1) can be written in a vector-matrix format as follows:

$$\frac{du(t)}{dt} = -Au(t) + W_0 g(u(t)) + \sum_{i=1}^{n} W_i g(u(t-\tau_i(t))) + U. \quad (2)$$

Throughout the paper, we need the following notation. Let $B^T, B^{-1}, \lambda_M(B), \lambda_m(B)$, and $\|B\| = \sqrt{\lambda_M(B^T B)}$ denote the transpose, the inverse, the largest eigenvalue, the smallest eigenvalue, and the Euclidean norm of a square matrix B, respectively. Let $B > 0 (B < 0)$ denote a positive (negative) definite symmetric matrix. Let I denote an identity matrix with compatible dimension. Let $a_m = \min(\underline{a}_i), E = (e_{ij})_{n\times n}$, where $e_{ii} = \overline{w}_{ii}^0$ if $i = j$ and $e_{ij} = \max(|\underline{w}_{ij}^0|, |\overline{w}_{ij}^0|)$ if $i \neq j$. Let $F = (f_{ij})_{n\times n}$, where $f_{ij} = \max(|\underline{w}_{ij}|, |\overline{w}_{ij}|)$. Let $F_k = (f_{ij}^k)_{n\times n}$, where $f_{ij}^k = \max(|\underline{w}_{ij}^k|, |\overline{w}_{ij}^k|)$, $k = 1, \ldots, n$. All the delays, either constant or time-varying delays, are bounded, and let $\rho = \max(\rho_i)$, where $\rho_i = \sup_{1\leq j\leq n}(\tau_{ij}(t)), i, j = 1, 2, \ldots, n$.

Remark 2.1: The following neural network models:

$$\dot{u}_i(t) = -a_i u_i(t) + \sum_{j=1}^{n} w_{ij}^0 g_j(u_j(t)) + \sum_{j=1}^{n} w_{ij} g_j(u_j(t-d_j(t))) + U_i \quad (3)$$

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$$\dot{u}_i(t) = -a_i u_i(t) + \sum_{j=1}^{n} w_{ij}^0 g_j(u_j(t)) + \sum_{j=1}^{n} w_{ij} g_j(u_j(t-h)) + U_i \quad (4)$$

where $d_j(t) > 0$ and $h > 0$, have been extensively studied in many literatures [1], [3], [11], [18]. The above two models can be expressed in the following compact vector-matrix form, respectively:

$$\frac{du(t)}{dt} = -Au(t) + W_0 g(u(t)) + Wg(u(t-d(t))) + U \quad (5)$$

and

$$\frac{du(t)}{dt} = -Au(t) + W_0 g(u(t)) + Wg(u(t-h)) + U \quad (6)$$

where $d(t) = (d_1(t), \ldots, d_n(t))^T$. To the best of our knowledge, the present stability results in the form of LMI are only intended for model (3) and (4), while no stability result for model (1) in the form of LMI has been reported yet. It is important to study model (1) using LMI technique as was done for models (3) and (4). This motivates us to present some robust stability criteria for model (1) via LMI technique.

Assumption 2.1: The bounded activation function, $g_j(u_j(t))$, satisfies $0 \leq (g_j(\xi) - g_j(\zeta))/(\xi - \zeta) \leq \delta_j$, for arbitrary $\xi, \zeta \in \Re, \xi \neq \zeta$, and for some positive constant $\delta_j > 0, j = 1, 2, \ldots, n$.

Let $\Delta = \mathrm{diag}(\delta_1, \delta_2, \ldots, \delta_n)$. Obviously, Δ is nonsingular.

Lemma 2.1: (see [23]) Let X and Y be two arbitrary real vectors with same dimensions, then for any matrices Π and $Q > 0$ with compatible dimensions, and any constants $m > 0$, $l > 0$, the following inequality holds:

$$-mX^T QX + 2lX^T \Pi Y \leq l^2 Y^T \Pi^T (mQ)^{-1} \Pi Y. \quad (7)$$

Lemma 2.2: (see [23]) For any square matrix X and identity matrix I with compatible dimensions, if $\|X\| < 1$, then

$$\|(I - X)^{-1}\| \leq \frac{1}{1 - \|X\|}. \quad (8)$$

III. Global Robust Exponential Stability Results

Here, we assume that system (2) has an equilibrium point $u^* = [u_1^*, \ldots, u_n^*]^T$ for a given U. Transformation $x(\cdot) = u(\cdot) - u^*$ converts the system (2) into the following form:

$$\frac{dx(t)}{dt} = -Ax(t) + W_0 f(x(t)) + \sum_{i=1}^{n} W_i f(x(t - \tau_i(t)))$$
$$x(t) = \phi(t), \quad t \in [-\rho, 0] \quad (9)$$

where $x(t) = (x_1(t), \ldots, x_n(t))^T$ is the state vector of the transformed system, $f(x(t)) = (f_1(x_1(t)), \ldots, f_n(x_n(t)))^T, f_j(x_j(t)) = g_j(x_j(t) + u_j^*) - g_j(u_j^*)$ with $f_j(0) = 0, \phi(t)$ is a continuous and bounded function with the upper bound $\bar{\phi} = \sup_{-\rho \leq \vartheta \leq 0} \|\phi(\vartheta)\|$. By Assumption 2.1, we can see that $f_j(x_j(t))x_j(t) \geq 0$ and $|f_j(x_j(t))| \leq \delta_j |x_j(t)|, j = 1, 2, \ldots, n$.

Clearly, dynamics at the equilibrium point u^* is globally exponentially stable for system (2) if and only if the zero solution of system (9) is globally exponentially stable.

Theorem 3.1: Under the conditions $0 \leq \dot{\tau}_{ij}(t) < 1$, if there exist diagonal matrices $P = \mathrm{diag}(p_1, \ldots, p_n) > 0$ and $Q_i = \mathrm{diag}(q_{i1}, \ldots, q_{\mathrm{in}}) > 0$, such that the following LMI holds:

$$\begin{bmatrix} \Theta & PF_1 & \ldots & PF_n \\ F_1^T P & -\eta_1 Q_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ F_n^T P & 0 & \ldots & -\eta_n Q_n \end{bmatrix} < 0 \quad (10)$$

where $\Theta = -2a_m P\Delta^{-1} + PE + E^T P + \sum_{i=1}^{n} Q_i$, and $\eta_i = \inf_{1 \leq j \leq n}(1 - \dot{\tau}_{ij}(t)), i, j = 1, \ldots, n$, then the equilibrium point of system (2) is globally robustly exponentially stable.

Proof: By Schur complement [2], (10) is equivalent to the following condition:

$$-2a_m P\Delta^{-1} + PE + E^T P + \sum_{i=1}^{n} \frac{1}{\eta_i} PF_i Q_i^{-1} F_i^T P + \sum_{i=1}^{n} Q_i < 0. \quad (11)$$

In view of the interval parameter ranges and in a similar manner to [19], from (11) we can deduce the following condition to hold:

$$-2a_m P\Delta^{-1} + PW_0 + W_0^T P + \sum_{i=1}^{n} \frac{1}{\eta_i} PW_i Q_i^{-1} W_i^T P + \sum_{i=1}^{n} Q_i < 0. \quad (12)$$

In the following, we will prove Theorem 3.1 in two steps. Firstly, we prove the uniqueness of the equilibrium point by contradiction. Consider the equilibrium equation of (9), i.e.,

$$0 = -Ax^* + W_0 f(x^*) + \sum_{i=1}^{n} W_i f(x^*). \quad (13)$$

It is evident that if $f(x^*) = 0$, then $x^* = 0$. Now suppose $f(x^*) \neq 0$, then multiply both sides of (13) by $2f^T(x^*)P$

$$0 = -2f^T(x^*)PAx^* + 2f^T(x^*)PW_0 f(x^*) + 2f^T(x^*)P\sum_{i=1}^{n} W_i f(x^*). \quad (14)$$

Because $f_j^2(x_j(t)) \leq \delta_j x_j f_j(x_j(t)), \forall x_j(t) \in \Re, j = 1, \ldots, n$, we have

$$f^T(x^*)PAx^* \geq a_m f^T(x^*)P\Delta^{-1} f(x^*). \quad (15)$$

Thus, by (15) and Lemma 2.1, from (14) we get

$$0 \leq f^T(x^*)\Big(-2a_m P\Delta^{-1} + PW_0 + W_0^T P + \sum_{i=1}^{n} \Big(\frac{1}{\eta_i} PW_i Q_i^{-1} W_i^T P + Q_i\Big)\Big) f(x^*). \quad (16)$$

On the other hand, from (12) we obtain

$$f^T(x^*)\Big(-2a_m P\Delta^{-1} + PW_0 + W_0^T P + \sum_{i=1}^{n} \Big(\frac{1}{\eta_i} PW_i Q_i^{-1} W_i^T P + Q_i\Big)\Big) f(x^*) < 0 \quad (17)$$

for $\forall f(x^*) \neq 0$.

Clearly, (17) contradicts with (16), which in turn implies that at the equilibrium point $x^*, f(x^*) = 0$, and also $x^* = 0$. This means that the origin of (9) or the equilibrium point u^* of model (2) is unique for a given U.

Secondly, we will prove the global exponential stability of the equilibrium point x^*. Consider the following Lyapunov–Krasovskii functional:

$$\begin{aligned} V(x(t)) = &(n+1)x^T(t)x(t)e^{kt} \\ &+ 2\alpha \sum_{i=1}^{n} p_i e^{kt} \int_0^{x_i(t)} f_i(s)ds \\ &+ \sum_{i=1}^{n}(\alpha+\beta_i) \\ &\times \sum_{j=1}^{n} \int_{t-\tau_{ij}(t)}^{t} e^{k(s+\rho_i)} q_{ij} f_j^2(x_j(s))ds. \end{aligned} \quad (18)$$

The time derivative of the functional (18) along the trajectories of system (9) is obtained as follows:

$$\begin{aligned} \dot{V}(x(t)) \le e^{kt} \Bigg\{ &(n+1)kx^T(t)x(t) \\ &+ 2(n+1)x^T(t)\Bigg(- Ax(t) \\ &+ W_0 f(x(t)) + \sum_{i=1}^{n} W_i f(x(t-\tau_i(t)))\Bigg) \\ &+ 2\alpha k \sum_{i=1}^{n} p_i \int_0^{x_i(t)} f_i(s)ds \\ &+ 2\alpha f^T(x(t))P\Bigg(- Ax(t) + W_0 f(x(t)) \\ &+ \sum_{i=1}^{n} W_i f(x(t-\tau_i(t)))\Bigg) \\ &+ \sum_{i=1}^{n}(\alpha+\beta_i)[f^T(x(t))e^{k\rho_i}Q_i f(x(t)) \\ &- \eta_i f^T(x(t-\tau_i(t)))Q_i f(x(t-\tau_i(t)))]\Bigg\}. \end{aligned} \quad (19)$$

Since

$$2\sum_{i=1}^{n} p_i \int_0^{x_i(t)} f_i(s)ds \le x^T(t)P\Delta x(t) \quad (20)$$

by (15) and (20), (19) becomes

$$\begin{aligned} \dot{V}(x(t)) \le\ e^{kt} \Bigg\{ &-(n+1)x^T(t) \\ &\times \left(2a_m I - \frac{\varepsilon}{\alpha} I - \frac{\varepsilon}{n+1} P\Delta\right) x(t) \\ &+ 2(n+1)x^T(t) \\ &\times \left(W_0 f(x(t)) + \sum_{i=1}^{n} W_i f(x(t-\tau_i(t)))\right) \\ &- 2a_m \alpha f^T(x(t))P\Delta^{-1} f(x(t)) + 2\alpha f^T(x(t))P \\ &\times \left(W_0 f(x(t)) + \sum_{i=1}^{n} W_i f(x(t-\tau_i(t)))\right) \\ &+ \sum_{i=1}^{n}(\alpha+\beta_i)(f^T(x(t))e^{\frac{\varepsilon}{\alpha}\rho_i} Q_i f(x(t)) \\ &- \eta_i f^T(x(t-\tau_i(t)))Q_i f(x(t-\tau_i(t))))\Bigg\} \end{aligned} \quad (21)$$

where $\varepsilon = k\alpha, \varepsilon > 0$ is a small positive constant.

In order to prove $\dot{V}(x(t)) < 0$, firstly, we choose β_i such that

$$\beta_i \ge \frac{(n+1)^2\|W_i\|^2}{2a_m\eta_i\lambda_m(Q_i)}, \qquad i = 1, 2, \ldots, n. \quad (22)$$

Secondly, we choose a sufficiently small constant $\varepsilon > 0$ and a sufficiently large constant $\alpha > 0$ such that

$$2a_m I - \frac{\varepsilon}{\alpha} I - \frac{\varepsilon}{n+1} P\Delta > 0 \quad (23)$$

$$\begin{aligned} \left\|\frac{\varepsilon}{2a_m\alpha} I\right\| + \frac{\varepsilon}{2a_m(n+1)}\|P\Delta\| \le &1 - \frac{(n+1)^2\|W_i\|^2}{2a_m\eta_i\beta_i\lambda_m(Q_i)} \\ &< 1. \end{aligned} \quad (24)$$

By Lemma 2.2, from (24), we can obtain

$$\beta_i \ge \frac{(n+1)^2 \left\|W_i^T(2a_m I - \frac{\varepsilon}{\alpha} I - \frac{\varepsilon}{n+1} P\Delta)^{-1} W_i\right\|}{\eta_i\lambda_m(Q_i)}. \quad (25)$$

Consequently, we have

$$\eta_i\beta_i Q_i \ge (n+1)^2 W_i^T \left(2a_m I - \frac{\varepsilon}{\alpha} I - \frac{\varepsilon}{n+1} P\Delta\right)^{-1} W_i. \quad (26)$$

Moreover, from (12), we have

$$\begin{aligned} \alpha\Bigg(&2a_m P\Delta^{-1} - PW_0 - W_0^T P - \sum_{i=1}^{n} \frac{1}{\eta_i} PW_i Q_i^{-1} W_i^T P \\ &- \sum_{i=1}^{n} e^{\frac{\varepsilon}{\alpha}\rho_i} Q_i\Bigg) - \sum_{i=1}^{n} \beta_i e^{\frac{\varepsilon}{\alpha}\rho_i} Q_i \\ &- (n+1)^2 W_0^T \left(2a_m I - \frac{\varepsilon}{\alpha} I - \frac{\varepsilon}{n+1} P\Delta\right)^{-1} W_0 > 0 \end{aligned} \quad (27)$$

for some sufficiently large constant $\alpha > 0$, sufficiently small constant $\varepsilon > 0$ and some fixed constants $\beta_i > 0, i = 1, 2, \ldots, n$.

By Lemma 2.1, we have

$$\begin{aligned} &- x^T(t)\left(2a_m I - \frac{\varepsilon}{\alpha} I - \frac{\varepsilon}{n+1} P\Delta\right) x(t) \\ &+ 2(n+1)x^T(t)W_0 f(x(t)) \\ &\le (n+1)^2 f^T(x(t)) W_0^T \Big(2a_m I - \frac{\varepsilon}{\alpha} I \\ &\quad - \frac{\varepsilon}{n+1} P\Delta\Big)^{-1} W_0 f(x(t)). \end{aligned} \quad (28)$$

Similarly, for $i = 1, 2, \ldots, n$

$$
\begin{aligned}
&- x^T(t)\left(2a_m I - \frac{\varepsilon}{\alpha} I - \frac{\varepsilon}{n+1} P\Delta\right) x(t) \\
&\quad + 2(n+1) x^T(t) W_i f(x(t-\tau_i(t))) \\
&\quad \le (n+1)^2 f^T(x(t-\tau_i(t))) W_i^T (2a_m I \\
&\qquad - \frac{\varepsilon}{\alpha} I - \frac{\varepsilon}{n+1} P\Delta)^{-1} W_i f(x(t-\tau_i(t)))
\end{aligned} \tag{29}
$$

and

$$
\begin{aligned}
&- \alpha\eta_i f^T(x(t-\tau_i(t))) Q_i f(x(t-\tau_i(t))) \\
&\quad + 2\alpha f^T(x(t)) P W_i f(x(t-\tau_i(t))) \\
&\quad \le \frac{\alpha}{\eta_i} f^T(x(t)) P W_i Q_i^{-1} W_i^T P f(x(t)).
\end{aligned} \tag{30}
$$

By (28)–(30), (21) becomes

$$
\begin{aligned}
&\dot{V}(x(t)) \\
&\le e^{kt} f^T(x(t)) \Bigg[(n+1)^2 W_0^T \left(2a_m I - \frac{\varepsilon}{\alpha} I \right.\\
&\quad \left. - \frac{\varepsilon}{n+1} P\Delta\right)^{-1} W_0 + \sum_{i=1}^{n} \frac{\alpha}{\eta_i} P W_i Q_i^{-1} W_i^T P \\
&\quad - 2\alpha a_m P\Delta^{-1} + \alpha\left(P W_0 + W_0^T P\right) \\
&\quad + \sum_{i=1}^{n} (\alpha + \beta_i) e^{\frac{\varepsilon}{\alpha}\rho_i} Q_i \Bigg] f(x(t)) \\
&\quad + e^{kt} \sum_{i=1}^{n} f^T(x(t-\tau_i(t))) \\
&\quad \times \Bigg[(n+1)^2 W_i^T \left(2a_m I - \frac{\varepsilon}{\alpha} I \right.\\
&\quad \left. - \frac{\varepsilon}{n+1} P\Delta\right)^{-1} W_i - \beta_i \eta_i Q_i \Bigg] f(x(t-\tau_i(t))).
\end{aligned} \tag{31}
$$

Considering (26) and (27) again, therefore, $\dot{V}(x(t)) < 0$ if $f(x(t)) \neq 0$ and $f(x(t-\tau_i(t))) \neq 0, i = 1, 2, \ldots, n$.

Note that $f(x(t)) \neq 0$ implies that $x(t) \neq 0$. In the similar way to the above procedure, we can show that for the two cases $f(x(t)) = 0, x(t) \neq 0$ and $f(x(t)) = x(t) = 0, \dot{V}(x(t)) < 0$. $\dot{V}(x(t)) = 0$ if and only if $f(x(t)) = x(t) = f(x(t-\tau_i(t))) = 0, i = 1, \ldots, n$.

Furthermore

$$
\begin{aligned}
V(x(0)) \le &\left\{ \sum_{i=1}^{n} \lambda_M(Q_i)(\alpha + \beta_i) \frac{e^{k\rho_i} - 1}{k} \right.\\
&\left. + (n+1) + \alpha\lambda_M(P\Delta) \right\} \overline{\phi}^2.
\end{aligned} \tag{32}
$$

Meanwhile, $V(x(t)) \ge (n+1) e^{kt} \|x(t)\|^2$, then

$$
\|x(t)\| \le \sqrt{\frac{\mathrm{Z}}{n+1}} \overline{\phi} e^{\frac{-k}{2} t} \tag{33}
$$

where $\mathrm{Z} = (n+1) + \alpha\lambda_M(P\Delta) + \sum_{i=1}^{n} \lambda_M(Q_i) (\alpha + \beta_i)(e^{k\rho_i} - 1/k)$.

Thus, the equilibrium point u^* of (2) is globally robustly exponentially stable. Summarizing the above, (12) is a sufficient condition for the global robust exponential stability of system (2). Furthermore, considering the interval parameter ranges, (10) is a worse case of (12). This completes the proof. ■

Theorem 3.2: Under the conditions $0 \le \dot{\tau}_{ij}(t) < 1$, if there exist positive constant $\theta > 0$ and diagonal matrices $Q_i = \mathrm{diag}(q_{i1}, \ldots, q_{\mathrm{in}}) > 0, i, j = 1, \ldots, n$, such that the following LMI holds:

$$
\begin{bmatrix}
\Theta & \theta F_1 & \cdots & \theta F_n \\
\theta F_1^T & -\eta_1 Q_1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\theta F_n^T & 0 & \cdots & -\eta_n Q_n
\end{bmatrix} < 0 \tag{34}
$$

where $\eta_i = \inf_{1\le j\le n}(1 - \dot{\tau}_{ij}(t)), \Theta = -2a_m\theta\Delta^{-1} + \theta\Phi + \sum_{i=1}^{n} Q_i, \Phi = (\phi_{ij})_{n\times n}, \phi_{ii} = 2\overline{w}_{ii}^0$ if $i = j$ and $\phi_{ij} = \max(|\overline{w}_{ij}^0 + \overline{w}_{ji}^0|, |\underline{w}_{ij}^0 + \underline{w}_{ji}^0|)$ if $i \neq j$, then the equilibrium point of system (2) is globally robustly exponentially stable.

Proof: Proof follows the steps of the proof of Theorem 3.1 by letting $P = \theta I$. ■

Remark 3.1: Theorem 3.2 includes θ and $Q_i, i = 1, \ldots, n$, which has less degrees of freedom than that in Theorem 3.1 (i.e., a single number θ instead of n numbers $p_1, p_2, \ldots, p_n$). However, Theorem 3.2 accounts for the sign difference of connection matrix, Theorem 3.1 fails to do so. Generally, Theorem 3.1 and Theorem 3.2 complement each other.

When $\tau_{ij}(t) = d_j(t)$ in (1), in which it has the same form as that in (3), we have the following results.

Corollary 3.1: Under the conditions $0 \le \dot{d}_j(t) < 1$, if there exist matrices $P = \mathrm{diag}(p_1, \ldots, p_n) > 0$ and $Q_1 = \mathrm{diag}(q_1, \ldots, q_n) > 0$, such that the following LMI holds:

$$
\begin{bmatrix}
-2a_m P\Delta^{-1} + PE + E^T P + Q_1 & PF \\
F^T P & -\eta_1 Q_1
\end{bmatrix} < 0 \tag{35}
$$

where $\eta_1 = \min(1 - \dot{d}_j(t)) > 0, j = 1, \ldots, n$, then the equilibrium point of neural networks (3) or (5) is globally robustly exponentially stable.

Corollary 3.2: Under the conditions $0 \le \dot{d}_j(t) < 1$, if there exist constant $\theta > 0$ and $Q_1 = \mathrm{diag}(q_1, \ldots, q_n) > 0$ such that the following LMI holds:

$$
\begin{bmatrix}
-2a_m\theta\Delta^{-1} + \theta\Phi + Q_1 & \theta F \\
\theta F^T & -\eta_1 Q_1
\end{bmatrix} < 0 \tag{36}
$$

where $\eta_1 = \min(1 - \dot{d}_j(t)), j = 1, \ldots, n, \Phi$ is defined in Theorem 3.2, then the equilibrium point of system (3) or (5) is globally robustly exponentially stable.

IV. Illustrative Examples

Example 4.1: Consider a two-state interval neural network defined by (1), $\tau_{11}(t) = (1 - e^{-t})/(1 + e^{-t}), \tau_{12}(t) = 0.5\tau_{11}(t), \tau_{21}(t) = 0.9\tau_{11}(t), \tau_{22}(t) =$

$0.75\tau_{11}(t), g_j(u_j(t)) = (|u_j(t)+1| - |u_j(t)-1|)/2, U_j = 1$, $j = 1,2$

$$\underline{W}_0 = (\underline{w}^0_{ij})_{2\times 2} = \begin{bmatrix} -2 & 1 \\ 0 & -0.1 \end{bmatrix}$$
$$\overline{W}_0 = (\overline{w}^0_{ij})_{2\times 2} = \begin{bmatrix} -1.5 & 1.21 \\ 0.01 & 0.2 \end{bmatrix}$$
$$\underline{W} = (\underline{w}_{ij})_{2\times 2} = \begin{bmatrix} -1.5 & -0.1 \\ 0 & -0.1 \end{bmatrix}$$
$$\overline{W} = (\overline{w}_{ij})_{2\times 2} = \begin{bmatrix} -1 & 0.16 \\ 0.05 & 0.016 \end{bmatrix}$$

and $\underline{A} = \overline{A} = \mathrm{diag}(4,4)$. Obviously, $0 \le \rho < 1$, and $\eta = 0.5$.

All the results in [3], [10], [14] cannot be applied to this example to ascertain the robust stability. Applying Theorem 3.1 of the present paper, we have

$$P = \begin{bmatrix} 7.9006 & 0 \\ 0 & 13.2996 \end{bmatrix}$$
$$Q_1 = \begin{bmatrix} 31.0903 & 0 \\ 0 & 35.4505 \end{bmatrix}$$
$$Q_2 = \begin{bmatrix} 31.0903 & 0 \\ 0 & 35.4505 \end{bmatrix}.$$

Therefore, the concerned neural network is globally robustly exponentially stable.

Example 4.2: We still consider the neural network as discussed in Example 4.1 except that $\tau_{ij}(t) = \tau$ is a single delay.

Pertaining to this example, [3, Th. 1] and [14, Th. 1] cannot ascertain the robust stability for this particular example.

Because

$$E = \begin{bmatrix} -1.5 & 1.21 \\ 0.01 & 0.2 \end{bmatrix} \quad F_1 = \begin{bmatrix} 1.5 & 0.16 \\ 0 & 0 \end{bmatrix} \quad F_2 = \begin{bmatrix} 0 & 0 \\ 0.05 & 0.1 \end{bmatrix}$$

and $a_m = 4$, applying Theorem 3.1 of this brief, we have

$$Q_1 = \begin{bmatrix} 35.8057 & 0 \\ 0 & 39.2137 \end{bmatrix}$$
$$P = \begin{bmatrix} 9.5840 & 0 \\ 0 & 5.6653 \end{bmatrix}$$
$$Q_2 = \begin{bmatrix} 35.8057 & 0 \\ 0 & 39.2137 \end{bmatrix}.$$

So the interval neural network is robustly exponentially stable.

V. Conclusion

Some new results on the global robust exponential stability of equilibrium point for recurrent neural network with multiple time-varying delays are presented, which are computationally efficient. Two illustrative examples are utilized to demonstrate the effectiveness of the proposed results.

References

[1] S. Arik, "An improved global stability result for delayed cellular neural networks," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 49, no. 8, pp. 1211–1214, Aug. 2002.

[2] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA: SIAM, 1994.

[3] J. Cao and J. Wang, "Global asymptotic and robust stability of recurrent neural networks with time delays," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 52, no. 2, pp. 417–426, Feb. 2005.

[4] J. Cao, H.-X. Li, and L. Han, "Novel results concerning global robust stability of delayed neural networks," *Nonlinear Anal. Ser. B*, vol. 7, no. 3, pp. 458–469, 2006.

[5] A. Chen, J. Cao, and L. Huang, "Global robust stability of interval cellular neural networks with time-varying delays," *Chaos, Solitons, Fractals*, vol. 23, pp. 787–799, 2005.

[6] T. Chen and L. Rong, "Robust global exponential stability of Cohen-Grossberg neural networks with time delays," *IEEE Trans. Neural Netw.*, vol. 15, no. 1, pp. 203–206, Jan. 2004.

[7] E. Fridman and U. Shaked, "Input-output approach to stability and L_2-gain analysis of systems with time-varying delays," *Syst. Contr. Lett.*, vol. 55, pp. 1041–1053, 2006.

[8] S. Hu and J. Wang, "Global asymptotic stability and global exponential stability of continuous-time recurrent neural networks," *IEEE Trans. Autom. Contr.*, vol. 47, no. 5, pp. 802–807, May 2002.

[9] X. Li and C. E. de Souza, "Criteria for robust stability and stabilization of uncertain linear systems with state delay," *Automatica*, vol. 33, no. 9, pp. 1657–1662, 2006.

[10] C. Li, X. Liao, and R. Zhang, "Global robust asymptotic stability of muiti-delayed interval neural networks: An LMI approach," *Phys. Lett. A*, vol. 328, pp. 452–462, 2004.

[11] X. Liao, G. Chen, and E. N. Sanchez, "Delay-dependent exponential stability analysis of delayed neural networks: An LMI approach," *Neural Netw.*, vol. 15, pp. 855–866, 2002.

[12] X. Liao, G. Chen, and E. N. Sanchez, "LMI-based approach for asymptotic stability analysis of delayed neural networks," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 49, no. 7, pp. 1033–1039, Jul. 2002.

[13] X. Liao and J. Wang, "Algebraic criteria for global exponential stability of celullar neural networks with multiple time delays," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 50, no. 2, pp. 268–275, Feb. 2003.

[14] X. Liao, K.-W. Wong, Z. Wu, and G. Chen, "Novel robust stability criteria for interval delayed Hopfield neural networks," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 48, no. 11, pp. 1355–1358, Nov. 2001.

[15] N. Ozcan and S. Arik, "Global robust stability analysis of neural networks with multiple time delays," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 53, no. 1, pp. 166–176, Jan. 2006.

[16] J. H. Park, "Robust stability of bidirectional associative memory neural networks with time delays," *Phys. Lett. A*, vol. 349, pp. 494–499, 2006.

[17] L. Rong, "LMI-based criteria for robust stability of Cohen-Grossberg neural networks with delay," *Phys. Lett. A*, vol. 339, pp. 63–73, 2005.

[18] T. Roska, C. W. Wu, M. Balsi, and L. O. Chua, "Stability and dynamics of delay-type general and cellular neural networks," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 39, no. 6, pp. 487–490, Jun. 1992.

[19] V. Singh, "Global robust stability of delayed neural networks: An LMI approach," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 52, no. 1, pp. 33–36, Jan. 2005.

[20] L. Xie, T. Liu, J. Liu, W. Gu, and S. Wong, "Robust stability for delayed neural networks with nonlinear perturbation," *Lect. Notes Comput. Sci.*, vol. 3496, pp. 203–208, 2005.

[21] K. Yuan, J. Cao, and H.-X. Li, "Robust stability of switched Cohen-Grossberg neural networks with mixed time-varying delays," *IEEE Trans. Syst., Man Cybern. B: Cybern.*, vol. 36, no. 6, pp. 1356–1363, Dec. 2006.

[22] Z. Zeng and J. Wang, "Improved conditions for global exponential stability of recurrent neural networks with time-varying delays," *IEEE Trans. Neural Netw.*, vol. 17, no. 3, pp. 623–635, Mar. 2006.

[23] H. Zhang, Z. Wang, and D. Liu, "Exponential stability analysis of neural networks with multiple time delays," *Lect. Notes Comput. Sci.*, vol. 3496, pp. 142–148, 2005.