Global Asymptotic Stability and Robust Stability of a Class of Cohen–Grossberg Neural Networks With Mixed Delays

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***Abstract*—This paper is concerned with the global asymptotic stability of a class of Cohen–Grossberg neural networks with both multiple time-varying delays and continuously distributed delays. Two classes of amplification functions are considered, and some sufficient stability criteria are established to ensure the global asymptotic stability of the concerned neural networks, which can be expressed in the form of linear matrix inequality and are easy to check. Furthermore, some sufficient conditions guaranteeing the global robust stability are also established in the case of parameter uncertainties.**

***Index Terms*—Cohen–Grossberg neural networks, distributed delays, global asymptotic stability, linear matrix inequality (LMI), multiple time-varying delays, nonnegative equilibrium points, robust stability.**

I. Introduction

Cohen–Grossberg [11] proposed in 1983 a neural network model described by the following system:

$$\dot{u}_i(t) = -a_i\left(u_i(t)\right)\left[c_i\left(u_i(t)\right) - \sum_{j=1}^{n} w_{ij}\tilde{g}_j\left(u_j(t)\right) + U_i\right] \quad (1)$$

where $u_i(t)$ is the state of neuron, $a_i(u_i(t))$ is an amplification function, $c_i(u_i(t))$ is a well-defined function to guarantee the existence of solution of system (1), $\tilde{g}_j(u_j(t))$ is an activation function describing the effects of input on the output of neuron, w_{ij} is the connection weight coefficient of the neural network, U_i is an external constant input, and $i,j = 1,\ldots,n$. System (1) includes a number of models from neurobiology, population biology, and evolution theory, as well as the Hopfield neural network model [20] as a special case. In electronic implementation of analog neural networks, delays always exist due to the transmission of signal and the finite switching speed of amplifiers. On the other hand, it is desirable to introduce delays into neural networks when dealing with problems associated with motions. Therefore, model (1) and its delayed version have attracted the attention of many researchers (see, e.g., [1], [5]–[7], [10], [19], [26], [27], [29], [33], [35], [36], [40], [42], [45]). Among them, [42] introduced constant delays into (1), which yields the following form:

$$\dot{u}_i(t) = -a_i(u_i(t))\left[c_i(u_i(t)) - \sum_{k=0}^{N}\sum_{j=1}^{n} w_{ij}^{k}\tilde{g}_j(u_j(t-\tau_k)) + U_i\right] \quad (2)$$

where $\tau_k > 0$ are bounded constant delays, w_{ij}^k are the connection weight coefficients, and other notations are the same as those in system (1), i.e., $k = 0,\ldots,N$ and $i,j = 1,\ldots,n$.

We remark that one can consider several types of delays in (2), [15]–[17], [31], and [32]. References [17] and [18] first investigated the stability problem of asymmetric Hopfield neural networks with continuously distributed delays. The characteristic of this kind of continuously distributed delays is that the delays range over the infinitely long duration. Dynamics of different kinds of neural networks with this kind of continuously distributed delays are widely studied [6], [8], [9], [25], [34]. Although the results of [6], [8], [9], [25], and [34] are generally easy to verify, all the results in [6], [8], [9], [25], and [34] take the absolute value operation on the connection weight coefficients. Therefore, the sign difference of entries in connection matrix is ignored, which leads to the ignorance of the neuron's excitatory and inhibitory effects on the neural network. However, some kinds of continuously distributed delays in a practical system often range over a finite duration. For example, one application of this kind of continuously distributed delay systems can be found in the modeling of feeding systems and combustion chambers in a liquid monopropellant rocket motor with pressure feeding [12], [13]. This kind of continuously distributed delay systems have been investigated in [22], [23], [38], and [41], but all the results in [22], [23], [38], and [41] are only suitable for the case of linear systems with constant delay. It is well known that neural networks are large-scale and complex nonlinear dynamic systems. Different types of delays may occur, and it is useful to investigate the dynamics of neural networks with different types of delays.

Motivated by the aforementioned discussion, the purpose of this paper is to establish sufficient conditions for the global asymptotic stability of the following system with both multiple

Manuscript received June 23, 2007; revised December 02, 2007 and June 12, 2008. First published August 04, 2008; current version published March 11, 2009. This work was supported in part by the National Natural Science Foundation of China under Grants 60534010, 60572070, 60728307, 60774048, and 60774093, by the 111 Project of the Ministry of Education of China under Grant B08015, by the Natural Science Foundation of Liaoning Province under Grant 20072025, by the Program for Cheung Kong Scholars and Innovative Research Groups of China under Grant 60521003, by the National High Technology Research and Development Program of China under Grant 2006AA04Z183, and by the Postdoctor Foundation of Northeastern University under Grant 20080314. This paper was recommended by Associate Editor M. Di Marco.

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Digital Object Identifier 10.1109/TCSI.2008.2002556

time-varying delays and a kind of continuously distributed delays ranging over a finite duration:

$$\dot{u}_i(t) = -a_i(u_i(t)) \times \left[c_i(u_i(t)) - \sum_{j=1}^{n} w_{ij}\tilde{g}_j(u_j(t)) - \sum_{k=1}^{N}\sum_{j=1}^{n} w_{ij}^k \tilde{f}_j(u_j(t-\tau_{kj}(t))) - \sum_{l=1}^{r}\sum_{j=1}^{n} b_{ij}^l \int_{t-d_l(t)}^{t} \tilde{h}_j(u_j(s))\,\mathrm{d}s + U_i \right] \quad (3)$$

where w_{ij}^k and b_{ij}^l are connection weight coefficients of the neural network, time delays $\tau_{kj}(t) > 0$ and $d_l(t) > 0$ are all bounded, U_i is an external constant input, $\tilde{f}_j(u_j(t))$ and $\tilde{h}_j(u_j(t))$ are activation functions, respectively, and other notations are the same as those in system (1), i.e., $i, j = 1, \ldots, n$, $k = 1, \ldots, N$, and $l = 1, \ldots, r$.

The neural network model studied here is more general and includes a large class of existing neural networks of the following two cases.

Case 1) $a_i(\varrho) > 0$ for all $\varrho \in \Re$ and $i = 1, 2, \ldots, n$.

Case 2) $a_i(\varrho) > 0$ for all $\varrho > 0$ and $a_i(0) = 0$, $i = 1, 2, \ldots, n$.

For example, in Case 1), model (3) contains the models studied in [3], [4], [7], [10], [19]–[21], [24], [26], [27], [33], [37], [42], [45]–[47], [49], and [50]. In Case 2), model (3) contains the classical Cohen–Grossberg neural model [11], [28], [30] and the famous Lotka–Volterra neural model [43], [44]. Instead of using the Jensen inequality approach, as used in [3] and [46], we construct a suitable Lyapunov–Krasovskii functional in this paper to deal with the continuously distributed delays, and some sufficient conditions are derived to ensure the global asymptotic stability of system (3) for Case 1) or 2), respectively, which can be expressed in the form of linear matrix inequality (LMI) and are independent of the values of time-varying delays and amplification functions. Corollaries are also given for some special cases of system (3). Moreover, a global robust stability criterion is also established for system (3) with parameter uncertainties.

Throughout this paper, let B^T, B^{-1}, $\lambda_m(B)$, $\lambda_M(B)$, and $\|B\| = \sqrt{\lambda_M(B^TB)}$ denote the transpose, the inverse, the smallest eigenvalue, the largest eigenvalue, and the Euclidean norm of a square matrix B, respectively. Let $B > 0\,(B < 0)$ denote a positive (negative) definite symmetric matrix. Let I and 0 denote the identity matrix and the zero matrix with compatible dimensions, respectively. Time delays $\tau_{ij}(t)$ and $d_l(t)$ are all bounded, i.e., $0 \le \tau_{ij}(t) \le \bar{\rho}$, $0 \le d_l(t) \le d_l^M$, $\rho = \max\{\bar{\rho}, d_l^M\}$, $\dot{\tau}_{ij}(t) \le \mu_{ij}$, $\dot{d}_l(t) \le \upsilon_l$, $\mu_{ij} \ge 0$, $\upsilon_l \ge 0$, $i = 1, \ldots, N$, $j = 1, \ldots, n$, and $l = 1, \ldots, r$.

II. Global Asymptotic Stability Results

In this section, we need the following assumptions and lemma.

Assumption 2.1: There exist constants $\gamma_i > 0$ such that the function $c_i(\cdot)$ satisfies

$$\frac{c_i(\zeta) - c_i(\xi)}{\zeta - \xi} \ge \gamma_i \quad (4)$$

for $\forall\, \zeta, \xi \in \Re$, and $\zeta \ne \xi$ $(i = 1, \ldots, n)$.

Assumption 2.2: The bounded activation functions $\tilde{g}_i(\cdot)$, $\tilde{f}_i(\cdot)$, and $\tilde{h}_i(\cdot)$ satisfy the following conditions:

$$0 \le \frac{\tilde{g}_i(\zeta) - \tilde{g}_i(\xi)}{\zeta - \xi} \le \delta_i^g \quad (5)$$

$$0 \le \frac{\tilde{f}_i(\zeta) - \tilde{f}_i(\xi)}{\zeta - \xi} \le \delta_i^f \quad (6)$$

$$0 \le \frac{\tilde{h}_i(\zeta) - \tilde{h}_i(\xi)}{\zeta - \xi} \le \delta_i^h \quad (7)$$

for $\forall\, \zeta, \xi \in \Re$, and $\zeta \ne \xi$ and for $\delta_i^g > 0$, $\delta_i^f > 0$, and $\delta_i^h > 0$ $(i = 1, \ldots, n)$.

Let $\Gamma = \mathrm{diag}(\gamma_1, \ldots, \gamma_n)$, $\Delta_g = \mathrm{diag}(\delta_1^g, \ldots, \delta_n^g)$, $\Delta_f = \mathrm{diag}(\delta_1^f, \ldots, \delta_n^f)$, and $\Delta_h = \mathrm{diag}(\delta_1^h, \ldots, \delta_n^h)$.

Assumption 2.3: The amplification function $a_i(\varrho)$ is positive and continuous, i.e., $a_i(\varrho) > 0$ for all $\varrho \in \Re$ $(i = 1, \ldots, n)$.

Lemma 2.1 (See [37]): Let X and Y be two real vectors with appropriate dimensions, and let Π and Q be two matrices with appropriate dimensions, where $Q > 0$. Then, for any two positive constants m and l, the following inequality holds:

$$-mX^TQX + 2lX^T\Pi Y \le l^2Y^T\Pi^T(mQ)^{-1}\Pi Y. \quad (8)$$

According to [35], for every external constant input U_i, neural network (3) has an equilibrium point $u^* = [u_1^*, \ldots, u_n^*]^{\mathrm{T}}$ if $a_i(\cdot)$, $c_i(\cdot)$, $\tilde{g}_i(\cdot)$, $\tilde{f}_i(\cdot)$, and $\tilde{h}_i(\cdot)$ satisfy the aforementioned conditions $(i = 1, \ldots, n)$. Letting $x_i(t) = u_i(t) - u_i^*$, then model (3) is transformed into the following form:

$$\dot{x}_i(t) = -A_i(x_i(t)) \left[C_i(x_i(t)) - \sum_{j=1}^{n} w_{ij} g_j(x_j(t)) - \sum_{k=1}^{N}\sum_{j=1}^{n} w_{ij}^k f_j(x_j(t-\tau_{kj}(t))) - \sum_{l=1}^{r}\sum_{j=1}^{n} b_{ij}^l \int_{t-d_l(t)}^{t} h_j(x_j(s))\,\mathrm{d}s \right] \quad (9)$$

or in a vector matrix form by the method in [48]

$$\dot{x}(t) = -A(x(t)) \left[C(x(t)) - Wg(x(t)) - \sum_{k=1}^{N} W_k f(x(t-\bar{\tau}_k(t))) - \sum_{l=1}^{r} B_l \int_{t-d_l(t)}^{t} h(x(s))\,\mathrm{d}s \right] \quad (10)$$

where $x(t) = (x_1(t), \ldots, x_n(t))^{\mathrm{T}}$, $A(x(t)) = \mathrm{diag}(A_1(x_1(t)), \ldots, A_n(x_n(t)))$, $A_i(x_i(t)) = a_i(x_i(t) + u_i^*)$, $g(x(t)) = (g_1(x_1(t)), \ldots, g_n(x_n(t)))^{\mathrm{T}}$, $g_i(x_i(t)) = \tilde{g}_i(x_i(t) + u_i^*) - \tilde{g}_i(u_i^*)$, $f(x(t)) =$

$(f_1(x_1(t)),\ldots,f_n(x_n(t)))^{\mathrm{T}}$, $f_i(x_i(t)) = \tilde{f}_i(x_i(t)+u_i^*) - \tilde{f}_i(u_i^*)$, $h(x(t)) = (h_1(x_1(t)),\ldots,h_n(x_n(t)))^{\mathrm{T}}$, $h_i(x_i(t)) = \tilde{h}_i(x_i(t)+u_i^*) - \tilde{h}_i(u_i^*)$, $C(x(t)) = (C_1(x_1(t)),\ldots,C_n(x_n(t)))^{\mathrm{T}}$, $C_i(x_i(t)) = c_i(x_i(t)+u_i^*) - c_i(u_i^*)$, $W=(w_{ij})_{n\times n}$, $f(x(t-\bar{\tau}_k(t))) = (f_1(x_1(t-\tau_{k1}(t))),\ldots,f_n(x_n(t-\tau_{kn}(t))))^{\mathrm{T}}$, $\bar{\tau}_k(t) = (\tau_{k1}(t),\ldots,\tau_{kn}(t))^{\mathrm{T}}$, $B_l = (b_{ij}^l)_{n\times n}$, $W_k = (w_{ij}^k)_{n\times n}$, $i,j=1,\ldots,n$, $k=1,\ldots,N$, and $l=1,\ldots,r$. The initial condition of (10) is of the form $x(\theta)=\varphi(\theta)$ for $-\rho\le\theta\le 0$, and its supremum bound is $\|\varphi\| = \sup_{-\rho\le\theta\le 0}\|\varphi(\theta)\|$.

We now state and prove the results of this paper as follows.

Theorem 2.1: Suppose that Assumptions 2.1–2.3 hold, $\dot{\tau}_{ij}(t)\le\mu_{ij}<1$, and $\dot{d}_l(t)\le\upsilon_l<1$. If there exist positive diagonal matrices P, D, Q_k, F, and M and positive definite symmetric matrices Y_l, H_l, $\bar{Y}_l$, $\bar{S}_l$, Y_P, S, F_k, and S_k such that the following conditions hold:

$$-\eta_k Q_k + F_k + S_k < 0 \quad (11)$$

$$\begin{aligned}\Omega_1 =& -2D\Gamma\Delta_f^{-1} + \sum_{k=1}^{N} Q_k \\ &+ \sum_{l=1}^{r} DB_l Y_l^{-1} B_l^T D/\bar{\upsilon}_l + DWY_P^{-1}W^T D \\ &+ \sum_{k=1}^{N} DW_k[\eta_k Q_k - F_k - S_k]^{-1} W_k^T D < 0 \end{aligned}\quad (12)$$

$$\begin{aligned}\Omega_2 =& \sum_{k=1}^{N} FW_k F_k^{-1} W_k^T F - 2F\Gamma\Delta_g^{-1} + FW + W^T F \\ &+ \sum_{l=1}^{r} FB_l \bar{Y}_l^{-1} B_l^T F/\bar{\upsilon}_l + Y_P + S < 0\end{aligned}\quad (13)$$

$$\begin{aligned}\Omega_3^1 =& -2M\Gamma\Delta_h^{-1} + MWS^{-1}W^T M \\ &+ \sum_{k=1}^{N} MW_k S_k^{-1} W_k^T M + \sum_{l=1}^{r} MB_l \bar{S}_l^{-1} B_l^T M/\bar{\upsilon}_l \\ &+ \sum_{l=1}^{r} \frac{\left(d_l^M\right)^2 (1+\bar{\upsilon}_l)}{2\bar{\upsilon}_l} H_l < 0\end{aligned}\quad (14)$$

$$-H_l + Y_l + \bar{Y}_l + \bar{S}_l < 0 \quad (15)$$

$$-2P\Gamma + \sum_{l=1}^{r} PB_l H_l^{-1} B_l^T P/\bar{\upsilon}_l < 0 \quad (16)$$

then the equilibrium point of model (10) is globally asymptotically stable, where $\eta_i = \min(1-\mu_{ij})$ and $\bar{\upsilon}_l = 1-\upsilon_l$ ($j=1,\ldots,n$, $l=1,\ldots,r$, and $i,k=1,\ldots,N$).

Proof: We will prove the theorem in two steps. First, we will show that (11)–(16) are sufficient conditions guaranteeing the uniqueness of the equilibrium point of system (10). Consider (10) at equilibrium point x^*, i.e.,

$$0 = -C(x^*) + Wg(x^*) + \sum_{k=1}^{N} W_k f(x^*) + \sum_{l=1}^{r} B_l d_l(t) h(x^*). \quad (17)$$

If $g(x^*)=0$, $f(x^*)=0$, and $h(x^*)=0$, then it is easy to see that $x^*=0$. Now, suppose that $x^*\ne 0$. Then, $g(x^*)\ne 0$, $f(x^*)\ne 0$, and $h(x^*)\ne 0$. Multiplying $2f^T(x^*)D, 2g^T(x^*)F$, and $2h^T(x^*)M$ on both sides of (17) yields

$$\begin{aligned}0 =& 2\left[f^T(x^*)D + 2g^T(x^*)F + 2h^T(x^*)M\right] \\ &\times\left[-C(x^*) + Wg(x^*) + \sum_{k=1}^{N} W_k f(x^*) \right.\\ &\left.+ \sum_{l=1}^{r} B_l d_l(t) h(x^*)\right] \\ \le& f^T(x^*)\Omega_1 f(x^*) + g^T(x^*)\Omega_2 g(x^*) + h^T(x^*)\Omega_3^1 h(x^*)\end{aligned}\quad (18)$$

where we have used Lemma 2.1.

However, for $g(x^*)\ne 0$, $f(x^*)\ne 0$, and $h(x^*)\ne 0$, from (12)–(14), we have

$$f^T(x^*)\Omega_1 f(x^*) + g^T(x^*)\Omega_2 g(x^*) + h^T(x^*)\Omega_3^1 h(x^*) < 0. \quad (19)$$

Obviously, (18) contradicts with (19), which, in turn, implies that, at the equilibrium point x^*, $g(x^*)=0$, $f(x^*)=0$, and $h(x^*)=0$. This means that the origin of (10) is a unique equilibrium point.

Second, we will show that conditions (11)–(16) are also sufficient conditions guaranteeing the global asymptotic stability of the equilibrium point of system (10). Consider the following Lyapunov–Krasovskii functional

$$V(x(t)) = V_1(x(t)) + V_2(x(t)) + V_3(x(t)) \quad (20)$$

where

$$V_1(x(t)) = \alpha\sum_{l=1}^{r}\left(\int_{t-d_l(t)}^{t}\left[\int_{s}^{t} h^T(x(\theta))\,\mathrm{d}\theta\right] H_l \left[\int_{s}^{t} h(x(\theta))\,\mathrm{d}\theta\right]\mathrm{d}s\right)$$

$$V_2(x(t)) = \alpha\sum_{l=1}^{r}\left(\int_{0}^{d_l(t)}\int_{t-s}^{t}(\theta-t+s)h^T(x(\theta))\,H_l h(x(\theta))\,\mathrm{d}\theta\,\mathrm{d}s\right)$$

$$\begin{aligned}V_3(x(t)) =& \sum_{i=1}^{N}(\alpha+\beta_i)\sum_{j=1}^{n}\int_{t-\tau_{ij}(t)}^{t} q_{ij} f_j^2(x_j(s))\,\mathrm{d}s \\ &+ 2\alpha\sum_{i=1}^{n}\bar{d}_i\int_0^{x_i(t)}\frac{f_i(s)}{A_i(s)}\,\mathrm{d}s \\ &+ 2(N+1)\sum_{i=1}^{n} p_i\int_0^{x_i(t)}\frac{s}{A_i(s)}\,\mathrm{d}s \\ &+ 2\alpha\sum_{i=1}^{n}\bar{f}_i\int_0^{x_i(t)}\frac{g_i(s)}{A_i(s)}\,\mathrm{d}s + 2\alpha\sum_{i=1}^{n}\bar{m}_i\int_0^{x_i(t)}\frac{h_i(s)}{A_i(s)}\,\mathrm{d}s\end{aligned}$$

where $H_l>0$, $q_{ij}>0$, $p_j>0$, $\bar{d}_j>0$, $\bar{f}_i>0$, $\bar{m}_j>0$, $\alpha>0$, and $\beta_i>0$ ($i=1,\ldots,N$ and $j=1,\ldots,n$).

The derivative of $V_1(x(t))$ along the trajectories of (10) is as follows:

$$\begin{aligned}\dot{V}_1(x(t)) &\le \alpha \sum_{l=1}^{r} \Bigg[-\bar{\upsilon}_l \left(\int_{t-d_l(t)}^{t} h^T(x(\theta))\mathrm{d}\theta \right) H_l \left(\int_{t-d_l(t)}^{t} h(x(\theta))\mathrm{d}\theta \right) \\ &\quad + 2 \int_{t-d_l(t)}^{t} (\theta - t + d_l(t)) h^T(x(\theta)) H_l h(x(t)) \mathrm{d}\theta \Bigg]. \end{aligned} \tag{21}$$

By Lemma 2.1, for $H_l > 0$ and positive constants $\bar{\upsilon}_l$, $l = 1, \ldots, r$, we have

$$2h^T(x(\theta)) H_l h(x(t)) \le \bar{\upsilon}_l h^T(x(\theta)) H_l h(x(\theta)) + \frac{h^T(x(t)) H_l h(x(t))}{\bar{\upsilon}_l}.$$

Then, from (21) it yields

$$\begin{aligned}\dot{V}_1(x(t)) &\le \alpha \sum_{l=1}^{r} \Bigg[-\bar{\upsilon}_l \left(\int_{t-d_l(t)}^{t} h^T(x(\theta))\mathrm{d}\theta \right) H_l \left(\int_{t-d_l(t)}^{t} h(x(\theta))\mathrm{d}\theta \right) \\ &\quad + \int_{t-d_l(t)}^{t} (\theta - t + d_l(t)) h^T(x(\theta)) H_l h(x(\theta)) \bar{\upsilon}_l \mathrm{d}\theta \\ &\quad + \frac{\left(d_l^M\right)^2}{2\bar{\upsilon}_l} h^T(x(t)) H_l h(x(t)) \Bigg]. \end{aligned} \tag{22}$$

Similarly, the derivative of $V_2(x(t))$ along the trajectories of (10) is as follows:

$$\begin{aligned}\dot{V}_2(x(t)) &\le \alpha \sum_{l=1}^{r} \Bigg[\frac{\left(d_l^M\right)^2}{2} h^T(x(t)) H_l h(x(t)) \\ &\quad - \int_{t-d_l(t)}^{t} (d_l(t) - t + \theta) h^T(x(\theta)) \\ &\quad \times H_l h(x(\theta)) \bar{\upsilon}_l d\theta \Bigg]. \end{aligned} \tag{23}$$

Thus, we have

$$\begin{aligned}\dot{V}_1(x(t)) + \dot{V}_2(x(t)) &\le \alpha \sum_{l=1}^{r} \left[\frac{\left(d_l^M\right)^2}{2} h^T(x(t)) H_l h(x(t)) (1 + 1/\bar{\upsilon}_l) \right] \\ &\quad - \alpha \sum_{l=1}^{r} \bar{\upsilon}_l \left(\int_{t-d_l(t)}^{t} h^T(x(\theta))\mathrm{d}\theta \right) H_l \left(\int_{t-d_l(t)}^{t} h(x(\theta))\mathrm{d}\theta \right). \end{aligned} \tag{24}$$

The derivative of $V_3(x(t))$ along the trajectories of (10) is given by (25), shown at the bottom of the page. where $P = \mathrm{diag}(p_1, \ldots, p_n)$, $F = \mathrm{diag}(\bar{f}_1, \ldots, \bar{f}_n)$, $M = \mathrm{diag}(\bar{m}_1, \ldots, \bar{m}_n)$, $Q_i = \mathrm{diag}(q_{i1}, \ldots, q_{\mathrm{in}})$, and $D = \mathrm{diag}(\bar{d}_1, \ldots, \bar{d}_n)$ $(i = 1, \ldots, N)$.

$$\begin{aligned}\dot{V}_3(x(t)) &\le 2(N+1)x^T(t)P \left[-C(x(t)) + Wg(x(t)) + \sum_{k=1}^{N} W_k f(x(t - \bar{\tau}_k(t))) + \sum_{l=1}^{r} B_l \int_{t-d_l(t)}^{t} h(x(s))\mathrm{d}s \right] \\ &\quad + \sum_{i=1}^{N} (\alpha + \beta_i) \left[f^T(x(t)) Q_i f(x(t)) - \eta_i f^T(x(t - \bar{\tau}_i(t))) Q_i f(x(t - \bar{\tau}_i(t))) \right] \\ &\quad + 2\alpha f^T(x(t)) D \left[-C(x(t)) + Wg(x(t)) + \sum_{k=1}^{N} W_k f(x(t - \bar{\tau}_k(t))) + \sum_{l=1}^{r} B_l \int_{t-d_l(t)}^{t} h(x(s))\mathrm{d}s \right] \\ &\quad + 2\alpha g^T(x(t)) F \left[-C(x(t)) + Wg(x(t)) + \sum_{k=1}^{N} W_k f(x(t - \bar{\tau}_k(t))) + \sum_{l=1}^{r} B_l \int_{t-d_l(t)}^{t} h(x(s))\mathrm{d}s \right] \\ &\quad + 2\alpha h^T(x(t)) M \left[-C(x(t)) + Wg(x(t)) + \sum_{k=1}^{N} W_k f(x(t - \bar{\tau}_k(t))) + \sum_{l=1}^{r} B_l \int_{t-d_l(t)}^{t} h(x(s))\mathrm{d}s \right] \end{aligned} \tag{25}$$

By Assumptions 2.1 and 2.2 and Lemma 2.1, we have (26), shown at the bottom of the page. Therefore, see (27), shown at the bottom of the next page. Now, we choose β_k such that

$$\beta_k \geq \frac{(N+1)^2\|PW_k\|^2\left\|\left(2P\Gamma-\sum_{l=1}^{r}PB_lH_l^{-1}B_l^TP/\bar{\upsilon}_l\right)^{-1}\right\|}{\eta_k\lambda_m(Q_k)}. \tag{28}$$

Then

$$(N+1)^2W_k^TP\left(2P\Gamma-\sum_{l=1}^{r}PB_lH_l^{-1}B_l^TP/\bar{\upsilon}_l\right)^{-1}PW_k-\beta_k\eta_kQ_k \leq 0, \qquad k=1,\ldots,N. \tag{29}$$

Meanwhile, see the second equation at the bottom of the next page, where Ω_2 is defined in (13). If we choose α such that

$$\alpha > \frac{\lambda_M\left[(N+1)^2W^TP\left(2P\Gamma-\sum_{l=1}^{r}PB_lH_l^{-1}B_l^TP/\bar{\upsilon}_l\right)^{-1}PW\right]}{\lambda_m(-\Omega_2)} \tag{30}$$

then we have $\Psi<0$ for $g(x(t))\neq 0$.

Similarly, for sufficiently large number $\alpha>0$, the condition in (31), shown at the bottom of page 622, hold for $f(x(t))\neq 0$, where Ω_1 is defined in (12).

$$\begin{aligned}
\dot{V}_3(x(t)) \leq & -(N+1)x^T(t)\left(2P\Gamma-\sum_{l=1}^{r}PB_lH_l^{-1}B_l^TP/\bar{\upsilon}_l\right)x(t)+2(N+1)x^T(t)PWg(x(t))\\
&+2(N+1)x^T(t)P\sum_{k=1}^{N}W_kf(x(t-\bar{\tau}_k(t)))+(N+1)\sum_{l=1}^{r}\bar{\upsilon}_l\left(\int_{t-d_l(t)}^{t}h^T(x(s))\,\mathrm{d}s\right)H_l\left(\int_{t-d_l(t)}^{t}h(x(s))\,\mathrm{d}s\right)\\
&+\sum_{i=1}^{N}(\alpha+\beta_i)\left[f^T(x(t))Q_if(x(t))-\eta_if^T(x(t-\bar{\tau}_i(t)))Q_if(x(t-\bar{\tau}_i(t)))\right]-2\alpha f^T(x(t))D\Gamma\Delta_f^{-1}f(x(t))\\
&+\alpha f^T(x(t))DWY_P^{-1}W^TDf(x(t))+\alpha g^T(x(t))Y_Pg(x(t))+2\alpha f^T(x(t))D\sum_{k=1}^{N}W_kf(x(t-\bar{\tau}_k(t)))\\
&+\alpha\sum_{l=1}^{r}f^T(x(t))DB_lY_l^{-1}B_l^TDf(x(t))/\bar{\upsilon}_l+\alpha\sum_{l=1}^{r}\bar{\upsilon}_l\left(\int_{t-d_l(t)}^{t}h^T(x(s))\,\mathrm{d}s\right)Y_l\left(\int_{t-d_l(t)}^{t}h(x(s))\,\mathrm{d}s\right)\\
&-2\alpha g^T(x(t))F\Gamma\Delta_g^{-1}g(x(t))+2\alpha g^T(x(t))FWg(x(t))+\alpha g^T(x(t))\sum_{k=1}^{N}FW_kF_k^{-1}W_k^TFg(x(t))\\
&+\alpha\sum_{k=1}^{N}f^T(x(t-\bar{\tau}_k(t)))F_kf(x(t-\bar{\tau}_k(t)))+\alpha\sum_{l=1}^{r}g^T(x(t))FB_l\bar{Y}_l^{-1}B_l^TFg(x(t))/\bar{\upsilon}_l\\
&+\alpha\sum_{l=1}^{r}\bar{\upsilon}_l\left(\int_{t-d_l(t)}^{t}h^T(x(s))\,\mathrm{d}s\right)\bar{Y}_l\left(\int_{t-d_l(t)}^{t}h(x(s))\,\mathrm{d}s\right)-2\alpha h^T(x(t))M\Gamma\Delta_h^{-1}h(x(t))\\
&+\alpha h^T(x(t))MWS^{-1}W^TMh(x(t))+\alpha g^T(x(t))Sg(x(t))+\alpha h^T(x(t))\sum_{k=1}^{N}MW_kS_k^{-1}W_k^TMh(x(t))\\
&+\alpha\sum_{k=1}^{N}f^T(x(t-\bar{\tau}_k(t)))S_kf(x(t-\bar{\tau}_k(t)))+\alpha\sum_{l=1}^{r}h^T(x(t))MB_l\bar{S}_l^{-1}B_l^TMh(x(t))/\bar{\upsilon}_l\\
&+\alpha\sum_{l=1}^{r}\bar{\upsilon}_l\left(\int_{t-d_l(t)}^{t}h^T(x(s))\,\mathrm{d}s\right)\bar{S}_l\left(\int_{t-d_l(t)}^{t}h(x(s))\,\mathrm{d}s\right).
\end{aligned} \tag{26}$$

Moreover, by condition (15) in Theorem 2.1, we have $H_l - Y_l - \bar{Y}_l - \bar{S}_l > 0$. Then, there exists a sufficiently large constant $\alpha > 0$ such that

$$\alpha(H_l - Y_l - \bar{Y}_l - \bar{S}_l) - (N+1)H_l > 0, \qquad k = 1, \ldots, r. \quad (32)$$

Therefore, by (14), (28), (30), (31), and (32), and from (27), we have $\dot{V}(x(t)) < 0$ for $g(x(t)) \neq 0$, $f(x(t)) \neq 0$, $h(x(t)) \neq 0$, and $f(x(t - \bar{\tau}_k(t))) \neq 0$. $\dot{V}(x(t)) = 0$ if and only if $f(x(t)) = 0$, $g(x(t)) = 0$, $h(x(t)) = 0$, and $f(x(t - \bar{\tau}_k(t))) = 0$. According to Lyapunov stability theory, the origin of (10) is globally asymptotically stable. ■

If we take another Lyapunov functional, we can obtain the following result.

Theorem 2.2: Suppose that Assumptions 2.1–2.3 hold, $\dot{\tau}_{ij}(t) \leq \mu_{ij} < 1$, and $\dot{d}_l(t) \leq \upsilon_l < 1$. If there exist positive diagonal matrices P, D, M, S, S_i^f, S^g, S_l^h, Q_i, and $H_l > 0$ such that the following LMI holds:

$$\tilde{\Xi} = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & \Omega_{15} & \Omega_{16} \\ * & \Omega_{22} & W^T M & W^T S & \Omega_{25} & \Omega_{26} \\ * & * & \Omega_{33} & 0 & \Omega_{35} & \Omega_{36} \\ * & * & * & \Omega_{44} & \Omega_{45} & \Omega_{46} \\ * & * & * & * & \Omega_{55} & 0 \\ * & * & * & * & * & \Omega_{66} \end{bmatrix} < 0 \quad (33)$$

then the equilibrium point of model (10) is globally asymptotically stable, where $\Omega_{11} = -2P\Gamma$, $\Omega_{12} = PW + S^g\Delta_g - D\Gamma$,

$$\begin{aligned}
\dot{V}(x(t)) \leq & \sum_{l=1}^{r} \bar{\upsilon}_l \left(\int_{t-d_l(t)}^{t} h^T(x(\theta))\,\mathrm{d}\theta \right) \left[\alpha(Y_l + \bar{Y}_l + \bar{S}_l - H_l) + (N+1)H_l\right] \left(\int_{t-d_l(t)}^{t} h(x(\theta))\,\mathrm{d}\theta \right) \\
& + f^T(x(t)) \Bigg[\sum_{k=1}^{N} \beta_k Q_k + \alpha \Bigg(\sum_{k=1}^{N} Q_k - 2D\Gamma\Delta_f^{-1} + \sum_{l=1}^{r} DB_l Y_l^{-1} B_l^T D / \bar{\upsilon}_l \\
& \qquad + DWY_P^{-1}W^T D + \sum_{k=1}^{N} DW_k[\eta_k Q_k - F_k - S_k]^{-1} W_k^T D \Bigg) \Bigg] f(x(t)) \\
& + g^T(x(t)) \Bigg[(N+1)^2 W^T P \left(2P\Gamma - \sum_{l=1}^{r} PB_l H_l^{-1} B_l^T P / \bar{\upsilon}_l \right)^{-1} PW \\
& \qquad + \alpha \left(\sum_{k=1}^{N} FW_k F_k^{-1} W_k^T F - 2F\Gamma\Delta_g^{-1} + FW + W^T F + \sum_{l=1}^{r} FB_l \bar{Y}_l^{-1} B_l^T F / \bar{\upsilon}_l + Y_P + S \right) \Bigg] g(x(t)) \\
& + \sum_{k=1}^{N} f^T(x(t - \bar{\tau}_k(t))) \left[(N+1)^2 W_k^T P \left(2P\Gamma - \sum_{l=1}^{r} PB_l H_l^{-1} B_l^T P / \bar{\upsilon}_l \right)^{-1} PW_k - \beta_k \eta_k Q_k \right] f(x(t - \bar{\tau}_k(t))) \\
& + \alpha h^T(x(t)) \Bigg[-2M\Gamma\Delta_h^{-1} + MWS^{-1}W^T M + \sum_{k=1}^{N} MW_k S_k^{-1} W_k^T M \\
& \qquad + \sum_{l=1}^{r} MB_l \bar{S}_l^{-1} B_l^T M / \bar{\upsilon}_l + \sum_{l=1}^{r} \frac{\left(d_l^M\right)^2 (1 + \bar{\upsilon}_l)}{2\bar{\upsilon}_l} H_l \Bigg] h(x(t)).
\end{aligned} \quad (27)$$

$$\begin{aligned}
\Psi = & g^T(x(t)) \Bigg[(N+1)^2 W^T P \left(2P\Gamma - \sum_{l=1}^{r} PB_l H_l^{-1} B_l^T P / \bar{\upsilon}_l \right)^{-1} PW \\
& + \alpha \left(\sum_{k=1}^{N} FW_k F_k^{-1} W_k^T F - 2F\Gamma\Delta_g^{-1} + FW + W^T F + \sum_{l=1}^{r} FB_l \bar{Y}_l^{-1} B_l^T F / \bar{\upsilon}_l + Y_P + S \right) \Bigg] g(x(t)) \\
\leq & \left[\lambda_M \left((N+1)^2 W^T P \left(2P\Gamma - \sum_{l=1}^{r} PB_l H_l^{-1} B_l^T P / \bar{\upsilon}_l \right)^{-1} PW \right) - \alpha \lambda_m(-\Omega_2) \right] \|g(x(t))\|^2
\end{aligned}$$

$\Omega_{13} = \sum_{i=1}^{N} S_i^f \Delta_f - M\Gamma$, $\Omega_{14} = \sum_{i=1}^{r} S_i^h \Delta_h - S\Gamma$, $\Omega_{15} = [PW_1, \ldots, PW_N]$, $\Omega_{16} = [PB_1, \ldots, PB_r]$, $\Omega_{22} = -2S^g + DW + W^T D$, $\Omega_{25} = [DW_1, \ldots, DW_N]$, $\Omega_{26} = [DB_1, \ldots, DB_r]$, $\Omega_{33} = \sum_{i=1}^{N}(Q_i - 2S_i^f)$, $\Omega_{35} = [MW_1, \ldots, MW_N]$, $\Omega_{36} = [MB_1, \ldots, MB_r]$, $\Omega_{45} = [SW_1, \ldots, SW_N]$, $\Omega_{44} = \sum_{l=1}^{r}[0.5(d_l^M)^2 H_l(1+1/\bar{v}_l) - 2S_l^h]$, $\Omega_{46} = [SB_1, \ldots, SB_r]$, $\Omega_{55} = \mathrm{diag}(-\eta_1 Q_1, \ldots, -\eta_N Q_N)$, $\Omega_{66} = \mathrm{diag}(-\bar{v}_1 H_1, \ldots, -\bar{v}_r H_r)$, $*$ denotes the corresponding symmetric part in a matrix, $\eta_i = \min(1-\mu_{ij})$, and $\bar{v}_l = 1 - v_l$ ($j = 1, \ldots, n$, $l = 1, \ldots, r$, and $i = 1, \ldots, N$).

Proof: Consider the following Lyapunov–Krasovskii functional:

$$V(x(t)) = V_1(x(t)) + V_2(x(t)) + V_4(x(t)) \tag{34}$$

where $V_1(x(t))$ and $V_2(x(t))$ are defined in (20), except that $\alpha = 1$, and

$$\begin{aligned} V_4(x(t)) = &\sum_{i=1}^{N}\sum_{j=1}^{n} \int_{t-\tau_{ij}(t)}^{t} q_{ij} f_j^2(x_j(s))\,\mathrm{d}s \\ &+ 2\sum_{i=1}^{n} \bar{m}_i \int_0^{x_i(t)} \frac{f_i(s)}{A_i(s)}\mathrm{d}s + 2\sum_{i=1}^{n} p_i \int_0^{x_i(t)} \frac{s}{A_i(s)}\mathrm{d}s \\ &+ 2\sum_{i=1}^{n} \bar{d}_i \int_0^{x_i(t)} \frac{g_i(s)}{A_i(s)}\mathrm{d}s + 2\sum_{i=1}^{n} \bar{s}_i \int_0^{x_i(t)} \frac{h_i(s)}{A_i(s)}\mathrm{d}s \end{aligned}$$

where $q_{ij} > 0$, $p_j > 0$, $\bar{d}_j > 0$, $\bar{m}_j > 0$, and $\bar{s}_j > 0$ ($i = 1, \ldots, N$ and $j = 1, \ldots, n$).

The derivatives of $V_4(x(t))$ along the trajectories of (10) is as (35), shown at the bottom of the page, where $P = \mathrm{diag}(p_1, \ldots, p_n)$, $Q_i = \mathrm{diag}(q_{i1}, \ldots, q_{\mathrm{in}})$, $D = \mathrm{diag}(\bar{d}_1, \ldots, \bar{d}_n)$, $M = \mathrm{diag}(\bar{m}_1, \ldots, \bar{m}_n)$, and $S = \mathrm{diag}(\bar{s}_1, \ldots, \bar{s}_n)$ ($i = 1, \ldots, N$).

By Assumption 2.2, the following inequalities hold for positive diagonal matrices S^g, S_i^f, and S_l^h ($i = 1, \ldots, N$ and $l = 1, \ldots, r$)

$$\begin{aligned} &2g^T(x(t))\,S^g\Delta_g x(t) - 2g^T(x(t))\,S^g g(x(t)) \geq 0 \\ &2f^T(x(t))\,S_i^f\Delta_f x(t) - 2f^T(x(t))\,S_i^f f(x(t)) \geq 0 \\ &2h^T(x(t))\,S_l^h\Delta_h x(t) - 2h^T(x(t))\,S_l^h h(x(t)) \geq 0. \end{aligned} \tag{36}$$

Combining (24), (35), and (36), we have

$$\dot{V}(x(t)) \leq \tilde{\phi}^T(t)\tilde{\Xi}\tilde{\phi}(t) \tag{37}$$

where $\tilde{\phi}^T(t) = [x^T(t) \;\; g^T(x(t)) \;\; f^T(x(t)) \;\; h^T(x(t)) \;\; f^T(x(t-\bar{\tau}_1(t))) \;\; \cdots \;\; f^T(x(t-\bar{\tau}_N(t))) \;\; \int_{t-d_1(t)}^{t} h^T(x(s))\mathrm{d}s \;\; \cdots \;\; \int_{t-d_r(t)}^{t} h^T(x(s))\mathrm{d}s]$.

Obviously, if $\tilde{\Xi} < 0$, then $\dot{V}(x(t)) < 0$ for $\forall\tilde{\phi}(t) \neq 0$. By Lyapunov stability theory, the origin of system (10) is globally asymptotically stable. ■

We have the following results, which can be derived in a similar manner to the proof of Theorem 2.2.

Theorem 2.3: Suppose that Assumptions 2.1–2.3 hold, $\dot{\tau}_{ij}(t) \leq \mu_{ij} < 1$, and $\dot{d}_l(t) \leq v_l < 1$. If there exist positive

$$\begin{aligned} f^T(x(t))\Bigg[\sum_{k=1}^{N}\beta_k Q_k + \alpha\Bigg(&\sum_{k=1}^{N} Q_k - 2D\Gamma\Delta_f^{-1} + \sum_{l=1}^{r} DB_l Y_l^{-1} B_l^T D/\bar{v}_l + DWY_P^{-1}W^T D \\ &+ \sum_{k=1}^{N} DW_k[\eta_k Q_k - F_k - S_k]^{-1} W_k^T D\Bigg)\Bigg] f(x(t)) \leq \left[\lambda_M\left(\sum_{k=1}^{N}\beta_k Q_k\right) - \alpha\lambda_m(-\Omega_1)\right]\|f(x(t))\|^2 < 0. \end{aligned} \tag{31}$$

$$\begin{aligned} \dot{V}_4(x(t)) \leq &-2x^T(t)P\Gamma x(t) + 2x^T(t)P\left[Wg(x(t)) + \sum_{k=1}^{N} W_k f(x(t-\bar{\tau}_k(t))) + \sum_{l=1}^{r} B_l \int_{t-d_l(t)}^{t} h(x(s))\,\mathrm{d}s\right] \\ &+ \sum_{i=1}^{N}\left[f^T(x(t))\,Q_i f(x(t)) - \eta_i f^T(x(t-\bar{\tau}_i(t)))\,Q_i f(x(t-\bar{\tau}_i(t)))\right] - 2g^T(x(t))\,D\Gamma x(t) \\ &+ 2g^T(x(t))\,D\left[Wg(x(t)) + \sum_{k=1}^{N} W_k f(x(t-\bar{\tau}_k(t))) + \sum_{l=1}^{r} B_l \int_{t-d_l(t)}^{t} h(x(s))\,\mathrm{d}s\right] - 2f^T(x(t))\,M\Gamma x(t) \\ &+ 2f^T(x(t))\,M\left[Wg(x(t)) + \sum_{k=1}^{N} W_k f(x(t-\bar{\tau}_k(t))) + \sum_{l=1}^{r} B_l \int_{t-d_l(t)}^{t} h(x(s))\,\mathrm{d}s\right] - 2h^T(x(t))\,S\Gamma x(t) \\ &+ 2h^T(x(t))\,S\left[Wg(x(t)) + \sum_{k=1}^{N} W_k f(x(t-\bar{\tau}_k(t))) + \sum_{l=1}^{r} B_l \int_{t-d_l(t)}^{t} h(x(s))\,\mathrm{d}s\right] \end{aligned} \tag{35}$$

diagonal matrices P, D, S^g, H_l, and Q_i such that the following LMI holds:

$$\tilde{\Xi}_1 = \begin{bmatrix} \tilde{\Omega}_{11} & \tilde{\Omega}_{12} & \tilde{\Omega}_{13} & \tilde{\Omega}_{14} \\ * & \tilde{\Omega}_{22} & \tilde{\Omega}_{23} & \tilde{\Omega}_{24} \\ * & * & \tilde{\Omega}_{33} & 0 \\ * & * & * & \tilde{\Omega}_{44} \end{bmatrix} < 0 \qquad (38)$$

then the equilibrium point of model (10) is globally asymptotically stable, where $\tilde{\Omega}_{11} = -2P\Gamma + \sum_{i=1}^{N} \Delta_f Q_i \Delta_f + \sum_{l=1}^{r}[0.5(d_l^M)^2 \Delta_h H_l \Delta_h (1 + 1/\bar{v}_l)]$, $\tilde{\Omega}_{12} = PW + S^g \Delta_g - D\Gamma$, $\tilde{\Omega}_{13} = [PW_1, \ldots, PW_N]$, $\tilde{\Omega}_{14} = [PB_1, \ldots, PB_r]$, $\tilde{\Omega}_{22} = -2S^g + DW + W^T D$, $\tilde{\Omega}_{23} = [DW_1, \ldots, DW_N]$, $\tilde{\Omega}_{24} = [DB_1, \ldots, DB_r]$, $\tilde{\Omega}_{33} = \mathrm{diag}(-\eta_1 Q_1, \ldots, -\eta_N Q_N)$, $\tilde{\Omega}_{44} = \mathrm{diag}(-\bar{v}_1 H_1, \ldots, -\bar{v}_r H_r)$, $*$ denotes the corresponding symmetric part in a matrix, $\eta_i = \min(1 - \mu_{ij})$, and $\bar{v}_l = 1 - v_l$ ($j = 1, \ldots, n, l = 1, \ldots, r$, and $i = 1, \ldots, N$).

Proof: Consider the following Lyapunov–Krasovskii functional

$$V(x(t)) = V_1(x(t)) + V_2(x(t)) + V_5(x(t)) \qquad (39)$$

where $V_1(x(t))$ and $V_2(x(t))$ are the same as those defined in (20), except that $\alpha = 1$, and

$$V_5(x(t)) = \sum_{i=1}^{N}\sum_{j=1}^{n} \int_{t-\tau_{ij}(t)}^{t} q_{ij} f_j^2(x_j(s))\,\mathrm{d}s + 2\sum_{i=1}^{n} \bar{d}_i \int_0^{x_i(t)} \frac{g_i(s)}{A_i(s)}\mathrm{d}s + 2\sum_{i=1}^{n} p_i \int_0^{x_i(t)} \frac{s}{A_i(s)}\mathrm{d}s.$$

In a similar manner to the proof of Theorem 2.2, Theorem 2.3 can easily be derived. ■

Theorem 2.4: Suppose that Assumptions 2.1–2.3 hold, $\dot{\tau}_{ij}(t) \le \mu_{ij} < 1$, and $\dot{d}_l(t) \le v_l < 1$. If there exist positive diagonal matrices P, Q, H_l, and Q_i such that the following LMI holds:

$$\hat{\Xi}_1 = \begin{bmatrix} \hat{\Omega}_{11} & \hat{\Omega}_{12} & \hat{\Omega}_{13} & \hat{\Omega}_{14} \\ * & \hat{\Omega}_{22} & 0 & 0 \\ * & * & \hat{\Omega}_{33} & 0 \\ * & * & * & \hat{\Omega}_{44} \end{bmatrix} < 0 \qquad (40)$$

then the equilibrium point of model (10) is globally asymptotically stable, where $\hat{\Omega}_{11} = -2P\Gamma + \sum_{i=1}^{N} \Delta_f Q_i \Delta_f + \Delta_g Q \Delta_g + \sum_{l=1}^{r}[0.5(d_l^M)^2 \Delta_h H_l \Delta_h (1 + 1/\bar{v}_l)]$, $\hat{\Omega}_{12} = PW$, $\hat{\Omega}_{13} = [PW_1, \ldots, PW_N]$, $\hat{\Omega}_{14} = [PB_1, \ldots, PB_r]$, $\hat{\Omega}_{22} = -Q$, $\hat{\Omega}_{33} = \mathrm{diag}(-\eta_1 Q_1, \ldots, -\eta_N Q_N)$, $\hat{\Omega}_{44} = \mathrm{diag}(-\bar{v}_1 H_1, \ldots, -\bar{v}_r H_r)$, $*$ denotes the corresponding symmetric part in a matrix, $\eta_i = \min(1 - \mu_{ij})$, and $\bar{v}_l = 1 - v_l$ ($j = 1, \ldots, n, l = 1, \ldots, r$, and $i = 1, \ldots, N$).

When $f(\cdot) = g(\cdot) = h(\cdot)$ and $d_l(t) = d_l$, i.e., $v_l = 0$, $l = 1, \ldots, r$, system (10) becomes

$$\dot{x}(t) = -A(x(t)) \left[C(x(t)) - Wf(x(t)) - \sum_{k=1}^{N} W_k f(x(t - \bar{\tau}_k(t))) - \sum_{l=1}^{r} B_l \int_{t-d_l}^{t} f(x(s))\,\mathrm{d}s \right]. \qquad (41)$$

We have the following results, which can be derived in a similar manner to the proof of Theorem 2.2.

Theorem 2.5: Suppose that Assumptions 2.1–2.3 hold and $\dot{\tau}_{ij}(t) \le \mu_{ij} < 1$. If there exist positive diagonal matrices D, Q_i, and $H_l > 0$ such that the following condition holds:

$$\sum_{i=1}^{N} \left(\frac{1}{\eta_i} DW_i Q_i^{-1} W_i^T D + Q_i \right) - 2D\Gamma\Delta_f^{-1} + DW + W^T D + \sum_{l=1}^{r} d_l^2 H_l + \sum_{l=1}^{r} DB_l H_l^{-1} B_l^T D < 0 \qquad (42)$$

then system (41) is globally asymptotically stable, where $\eta_i = \min(1 - \mu_{ij})$ ($j = 1, \ldots, n, l = 1, \ldots, r$, and $i = 1, \ldots, N$).

Corollary 2.1: Suppose that Assumptions 2.1–2.3 hold and $\dot{\tau}_{ij}(t) \le \mu_{ij} < 1$. If there exist positive diagonal matrices D, Q_i, and $Y_l > 0$ such that the following condition holds:

$$\sum_{i=1}^{N} \left(\frac{1}{\eta_i} DW_i Q_i^{-1} W_i^T D + Q_i \right) - 2D\Gamma\Delta_f^{-1} + DW + W^T D + \sum_{l=1}^{r} d_l Y_l + \sum_{l=1}^{r} d_l DB_l Y_l^{-1} B_l^T D < 0 \qquad (43)$$

then system (41) is globally asymptotically stable, where $\eta_i = \min(1 - \mu_{ij})$ ($j = 1, \ldots, n, l = 1, \ldots, r$, and $i = 1, \ldots, N$).

For neural network (41) with $b_{kj}^l = 0$ ($k, j = 1, \ldots, n$ and $l = 1, \ldots, r$), we have the following result, which can be obtained directly from Theorem 2.5.

Corollary 2.2: Suppose that Assumptions 2.1–2.3 hold and $\dot{\tau}_{ij}(t) \le \mu_{ij} < 1$. If there exist positive diagonal matrices D and Q_i such that the following condition holds:

$$\sum_{i=1}^{N} \left(\frac{1}{\eta_i} DW_i Q_i^{-1} W_i^T D + Q_i \right) - 2D\Gamma\Delta_f^{-1} + DW + W^T D < 0 \qquad (44)$$

then the equilibrium point of model (41) with $b_{kj}^l = 0$ is globally asymptotically stable, where $\eta_i = \min(1 - \mu_{ij})$ ($k, j = 1, \ldots, n, l = 1, \ldots, r$, and $i = 1, \ldots, N$).

For the following Cohen–Grossberg neural networks:

$$\dot{u}_i(t) = -a_i(u_i(t)) \left[c_i(u_i(t)) - \sum_{j=1}^{n} w_{ij} g_j(u_j(t)) - \sum_{j=1}^{n} w_{ij}^1 g_j(u_j(t - \tau_{ij}(t))) + U_i \right] \qquad (45)$$

where $a_i(u_i(t))$, $c_i(u_i(t))$, w_{ij}, w_{ij}^1, and $g_j(u_j(t))$ are the same as those in (3) ($i, j = 1, \ldots, n$). We have the following result.

Corollary 2.3: Suppose that Assumptions 2.1–2.3 hold and $\dot{\tau}_{ij}(t) \le \mu_{ij} < 1$. If there exist positive diagonal matrices D and Q_i such that the following condition holds:

$$\sum_{i=1}^{n} \left(\frac{1}{\eta_i} DE_i Q_i^{-1} E_i^T D + Q_i \right) - 2D\Gamma\Delta_g^{-1} + DW + W^T D < 0 \qquad (46)$$

then the equilibrium point of model (45) is globally asymptotically stable for a given U_i, where $\eta_i = \min(1 - \mu_{ij})$, E_k is a square matrix whose kth row is composed by the kth row of square matrix $W_1 = (w_{ij}^1)_{n \times n}$, and the other rows are all zero ($i, j, k = 1, \ldots, n$).

III. Global Robust Stability Result

Consider the Cohen–Grossberg neural network (9) with uncertainties in the form of

$$\begin{aligned}\dot{x}_i(t) = &- A_i\left(x_i(t)\right)\Bigg[C_i\left(x_i(t)\right) - \sum_{j=1}^{n}\left(w_{ij}+\delta w_{ij}(t)\right)\\ &\times g_j\left(x_j(t)\right) - \sum_{k=1}^{N}\sum_{j=1}^{n}\left(w_{ij}^{k}+\delta w_{ij}^{k}(t)\right)\\ &\times f_j\left(x_j\left(t-\tau_{kj}(t)\right)\right)\\ &-\sum_{l=1}^{r}\sum_{j=1}^{n}\left(b_{ij}^{l}+\delta b_{ij}^{l}(t)\right)\\ &\times\int_{t-d_l(t)}^{t} h_j\left(x_j(s)\right)\mathrm{d}s\Bigg] \end{aligned}\tag{47}$$

or in a vector matrix form

$$\begin{aligned}\dot{x}(t) = &- A\left(x(t)\right)\Bigg[C\left(x(t)\right) - \bar{W}g\left(x(t)\right)\\ &-\sum_{k=1}^{N}\bar{W}_k f(x\left(t-\bar{\tau}_k(t)\right))\\ &-\sum_{l=1}^{r}\bar{B}_l\int_{t-d_l(t)}^{t} h\left(x(s)\right)\mathrm{d}s\Bigg]\end{aligned}\tag{48}$$

where $\delta w_{ij}(t)$, $\delta w_{ij}^k(t)$, and $\delta b_{ij}^l(t)$ denote the unknown connection weight coefficients representing time-varying parameter uncertainties, and other notations are the same as those in (9). For convenience of description, we let $\delta W(t) = (\delta w_{ij}(t))_{n\times n}$, $\delta W_k(t) = (\delta w_{ij}^k(t))_{n\times n}$, $\delta B_l(t) = (\delta b_{ij}^l(t))_{n\times n}$, $\bar{W} = W + \delta W(t)$, $\bar{W}_k = W_k + \delta W_k(t)$, and $\bar{B}_l = B_l + \delta B_l(t)$ $(i, j = 1,\dots,n$, $l = 1,\dots,r$, and $k = 1,\dots,N)$.

Assumption 3.1: Parameter uncertainties satisfy $\delta W(t) = M_0F(t)G_0$, $\delta W_k(t) = M_kF(t)G_k$, and $\delta B_l(t) = \bar{M}_lF(t)\bar{G}_l$, respectively, where $M_0, G_0, M_k, G_k, \bar{M}_l$, and $\bar{G}_l$ are the known structural matrices and $F(t)$ is an unknown time-varying function matrix satisfying $F^T(t)F(t) \le I$ $(k = 1,\dots,N$ and $l = 1,\dots,r)$.

Lemma 3.1 (See [39]): If Y, $F(t)$, and Z are real matrices of appropriate dimensions with Λ satisfying $\Lambda = \Lambda^T$, then $\Lambda + YF(t)Z + (YF(t)Z)^T < 0$ for all $F^T(t)F(t) \le I$, if and only if there exist a positive constant $\varepsilon > 0$, such that $\Lambda + \varepsilon^{-1}YY^T + \varepsilon Z^TZ < 0$.

Theorem 3.1: Suppose that Assumptions 2.1–2.3 and 3.1 hold, $\dot{\tau}_{ij}(t) \le \mu_{ij} < 1$, and $\dot{d}_l(t) \le \upsilon_l < 1$. If there exist positive diagonal matrices D, Q_i, S, P, M, S_i^f, S^g, S_l^h, and $H_l > 0$, positive constants $\varepsilon_0 > 0$, $\varepsilon_i > 0$, $\gamma > 0$, and $\bar{\varepsilon}_l > 0$, $i = 1,\dots,N$, $l = 1,\dots,r$, $j = 1,\dots,n$, such that the following LMI holds:

$$\begin{bmatrix}\Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} & \Omega_{15} & \Omega_{16} & \Omega_{17} & \Omega_{18}\\ * & \Xi_{22} & W^TM & W^TS & \Omega_{25} & \Omega_{26} & \Omega_{27} & \Omega_{28}\\ * & * & \Omega_{33} & 0 & \Omega_{35} & \Omega_{36} & \Omega_{37} & \Omega_{38}\\ * & * & * & \Omega_{44} & \Omega_{45} & \Omega_{46} & \Omega_{47} & \Omega_{48}\\ * & * & * & * & \Xi_{55} & 0 & 0 & 0\\ * & * & * & * & * & \Xi_{66} & 0 & 0\\ * & * & * & * & * & * & \Omega_{77} & 0\\ * & * & * & * & * & * & * & \Omega_{88}\end{bmatrix} < 0 \tag{49}$$

then system (47) is globally robustly stable, where $\Omega_{17} = [PM_0, PM_1,\dots,PM_N, 0]$, $\Omega_{18} = [P\bar{M}_1,\dots,P\bar{M}_r]$, $\Xi_{22} = \Omega_{22} + (\varepsilon_0 + \gamma)G_0^TG_0$, $\Omega_{27} = [DM_0, DM_1,\dots,DM_N, 0]$, $\Omega_{28} = [D\bar{M}_1,\dots,D\bar{M}_r]$, $\Omega_{37} = [0, MM_1,\dots,MM_N, MM_0]$, $\Omega_{38} = [M\bar{M}_1,\dots,M\bar{M}_r]$, $\Omega_{47} = [0, SM_1,\dots,SM_N, SM_0]$, $\Omega_{48} = [S\bar{M}_1,\dots,S\bar{M}_r]$, $\Xi_{55} = \mathrm{diag}(-\eta_1Q_1 + \varepsilon_1G_1^TG_1,\dots,-\eta_NQ_N + \varepsilon_NG_N^TG_N)$, $\Xi_{66} = \mathrm{diag}(-\bar{\upsilon}_1H_1 + \bar{\varepsilon}_1\bar{G}_1^T\bar{G}_1,\dots,-\bar{\upsilon}_rH_r + \bar{\varepsilon}_r\bar{G}_r^TG_r)$, $\Xi_{77} = \mathrm{diag}(-\varepsilon_0I, -\varepsilon_1I,\dots,-\varepsilon_NI, -\gamma I)$, $\Xi_{88} = \mathrm{diag}(-\bar{\varepsilon}_1I,\dots,-\bar{\varepsilon}_rI)$, $*$ denotes the corresponding symmetric part in a matrix, and others are defined in Theorem 2.2.

Proof: Consider the same Lyapunov–Krasovskii functional defined in (20). In a similar manner to the proof of Theorem 2.2, the derivative of (20) along the trajectories of (48) is of the form

$$\dot{V}\left(x(t)\right) \le \tilde{\phi}^T(t)\bar{\Xi}\tilde{\phi}(t) \tag{50}$$

with $\tilde{\phi}(t)$ being defined in (37)

$$\bar{\Xi} = \begin{bmatrix}\Omega_{11} & \bar{\Omega}_{12} & \Omega_{13} & \Omega_{14} & \bar{\Omega}_{15} & \bar{\Omega}_{16}\\ * & \bar{\Omega}_{22} & \bar{\Omega}_{23} & \bar{\Omega}_{24} & \bar{\Omega}_{25} & \bar{\Omega}_{26}\\ * & * & \Omega_{33} & 0 & \bar{\Omega}_{35} & \bar{\Omega}_{36}\\ * & * & * & \Omega_{44} & \bar{\Omega}_{45} & \bar{\Omega}_{46}\\ * & * & * & * & \Omega_{55} & 0\\ * & * & * & * & * & \Omega_{66}\end{bmatrix} < 0 \tag{51}$$

where $\bar{\Omega}_{12} = P\bar{W} + S^g\Delta_g - D\Gamma$, $\bar{\Omega}_{15} = [P\bar{W}_1,\dots,P\bar{W}_N]$, $\bar{\Omega}_{16} = [P\bar{B}_1,\dots,P\bar{B}_r]$, $\bar{\Omega}_{22} = -2S^g + D\bar{W} + \bar{W}^TD$, $\bar{\Omega}_{23} = \bar{W}^TM$, $\bar{\Omega}_{24} = \bar{W}^TS$, $\bar{\Omega}_{25} = [D\bar{W}_1,\dots,D\bar{W}_N]$, $\bar{\Omega}_{26} = [D\bar{B}_1,\dots,D\bar{B}_r]$, $\bar{\Omega}_{35} = [M\bar{W}_1,\dots,M\bar{W}_N]$, $\bar{\Omega}_{36} = [M\bar{B}_1,\dots,M\bar{B}_r]$, $\bar{\Omega}_{45} = [S\bar{W}_1,\dots,S\bar{W}_N]$, $\bar{\Omega}_{46} = [S\bar{B}_1,\dots,S\bar{B}_r]$, and others are defined in Theorem 2.2.

According to Assumption 3.1, Lemma 3.1, and the Schur complement, in a similar routine to the proof in [46], (51) is equivalent to (49). The details are omitted. ■

IV. Global Asymptotic Stability of Cohen–Grossberg Neural Networks With Nonnegative Equilibrium Points

In the original paper of Cohen–Grossberg [11], the Cohen–Grossberg neural network model was proposed as a kind of competitive-cooperation dynamical system for decision rules, pattern formation, and parallel memory storage. Therefore, each state of the neuron might be the population size, activity, or concentration of the ith species in the system, which is always nonnegative. It is clear that this subset of Cohen–Grossberg neural networks includes the famous Lotka–Volterra recurrent neural networks [11], [43], [44].

When systems (1) and (3) are applied to biology, initial conditions will be of the following type:

$$u_i(t) = \phi_i(t) \geq 0, \qquad -\rho \leq t \leq 0$$
$$u_i(0) = \phi_i(0) > 0, \qquad i \in \Re \tag{52}$$

where each $\phi_i(\cdot)$ is a continuous function defined on $[-\rho, 0]$.

In applications, the activation functions of model (3) may not be bounded, for example, in the Lotka–Volterra model [43], [44]. Therefore, in this section, we will employ the following assumptions and lemmas.

Assumption 4.1: $c_i(\cdot)$ satisfies (4) and $c_i(0) = 0$, $i = 1, \ldots, n$.

Assumption 4.2: Activation functions $\tilde{g}_i(\cdot)$, $\tilde{f}_i(\cdot)$, and $\tilde{h}_i(\cdot)$ satisfy conditions (5)–(7), and $\tilde{g}_i(0) = 0$, $\tilde{f}_i(0) = 0$, and $\tilde{h}_i(0) = 0$, $i = 1, \ldots, n$.

Assumption 4.3: The amplification function $a_i(\varrho) > 0$ for all $\varrho > 0$ and $a_i(0) = 0$, and for any $\epsilon > 0$, $\int_0^\epsilon (d\varrho / a_i(\varrho)) = +\infty$ holds for all $i = 1, \ldots, n$.

Lemma 4.1: Assume that $a_i(\varrho)$ satisfies Assumption 4.3. Then, the solution of the system (3) is positive with the initial condition (52).

If Assumption 4.3 holds, then the nonnegative equilibrium point of system (3) is a solution of the following:

$$u_i(F_i(u_i) + U_i) = 0, \qquad i = 1, \ldots, n \tag{53}$$

where

$$F_i(u_i) = c_i(u_i) - \sum_{j=1}^{n} w_{ij}\tilde{g}_j(u_j) - \sum_{k=1}^{N}\sum_{j=1}^{n} w_{ij}^k \tilde{f}_j(u_j) - \sum_{l=1}^{r}\sum_{j=1}^{n} b_{ij}^l \int_{t-d_l(t)}^{t} \tilde{h}_j(u_j(s))\,\mathrm{d}s.$$

Although (53) might possess multiple solutions, in a similar routine to [30Proof of Proposition 1], we can show that if u_i^* is an asymptotically stable nonnegative equilibrium point of system (3), then it must be a solution of the following problem:

$$u_i^* \geq 0 \quad F_i(u_i^*) + U_i \geq 0 \quad u_i^*(F_i(u_i^*) + U_i) = 0 \tag{54}$$

where $i = 1, \ldots, n$.

Lemma 4.2 (See [30]): Equation (54) has a unique solution for every U_i if and only if $\overline{F}(u)$ is norm coercive, i.e.,

$$\lim_{\|u\| \to \infty} \left\|\overline{F}(u)\right\| = \infty$$

and locally univalent, where $\overline{F}(u) = F(u^+) + u^-$, $F(u) = (F_1(u_1), \ldots, F_n(u_n))^{\mathrm{T}}$, $u^+ = (u_1^+, u_2^+, \ldots, u_n^+)^{\mathrm{T}}$, $u^- = (u_1^-, u_2^-, \ldots, u_n^-)^{\mathrm{T}}$, $u_i^+ = u_i$ if $u_i \geq 0$ and $u_i^+ = 0$ if $u_i < 0$, $u_i^- = u_i$ if $u_i \leq 0$ and $u_i^- = 0$ if $u_i > 0$, $i = 1, \ldots, n$.

Lemma 4.3 (See [14], [30]): Let T be an $n \times n$ matrix, $D = \mathrm{diag}(D_1, \ldots, D_n)$, and $G = \mathrm{diag}(G_1, \ldots, G_n)$. If there exists a positive definite diagonal matrix $P = \mathrm{diag}(P_1, \ldots, P_n)$ such that

$$P(D - TG) + (D - TG)^{\mathrm{T}} P > 0$$

holds, then, for any positive definite diagonal matrix $\overline{D} \geq D$ and nonnegative definite diagonal matrix K satisfying $0 \leq K \leq G$, we have $\det(\overline{D} - TK) \neq 0$, i.e., $\overline{D} - TK$ is nonsingular.

Theorem 4.1: Suppose that Assumptions 4.1–4.3 hold, $\dot{\tau}_{ij}(t) \leq \mu_{ij}$, and $\dot{d}_l(t) \leq \upsilon_l$ ($0 \leq \mu_{ij} < 1$ and $0 \leq \upsilon_l < 1$). Then, system (3) with initial conditions (52) has a unique nonnegative equilibrium point, which is globally asymptotically stable if there exist positive definite diagonal matrices P, Q, Q_k, and H_l such that the following condition holds:

$$\Xi = \begin{bmatrix} \Theta_1 & -(PW\Delta_g)^T & -(PW\Delta_g)^T \\ * & \Theta_2 & \Theta_3 \\ * & * & \Theta_4 \end{bmatrix} > 0 \tag{55}$$

where $*$ denotes the symmetric parts in a matrix, and its variables are described

$$\begin{aligned} \Theta_1 = {} & 2P\Gamma - PWQ^{-1}W^TP - \Delta_g Q \Delta_g \\ & - \sum_{k=1}^{N} \left(PW_kQ_k^{-1}W_k^TP/\eta_k + \Delta_f Q_k \Delta_f\right) \\ & - \sum_{l=1}^{r} \Big[PB_lH_l^{-1}B_l^TP\overline{\upsilon}_l \\ & \qquad + 0.5\left(d_l^M\right)^2 (1 + 1/\overline{\upsilon}_l)\Delta_h H_l \Delta_h\Big] \\ \Theta_2 = {} & 2P\Gamma - \sum_{k=1}^{N}\left(\Delta_f Q_k \Delta_f + PW_k\Delta_f + (PW_k\Delta_f)^{\mathrm{T}}\right) \\ \Theta_3 = {} & -\sum_{l=1}^{r} PB_l\Delta_h - \sum_{k=1}^{N} (PW_k\Delta_f)^{\mathrm{T}} \\ \Theta_4 = {} & 2P\Gamma - \sum_{l=1}^{r}\left(\Delta_h H_l \Delta_h + PB_l\Delta_h + (PB_l\Delta_h)^{\mathrm{T}}\right) \\ \eta_i = {} & \min(1 - \mu_{ij}) \quad \overline{\upsilon}_l = (1 - \upsilon_l). \end{aligned}$$

Proof: We prove Theorem 4.1 in two steps. First, we will show that (55) ensures the existence and uniqueness of a nonnegative equilibrium point of system (3).

From (55), we know that $\Theta_1 > 0$, which also means that

$$A^S = P\left(\Gamma - W\Delta_g - \sum_{k=1}^{N} W_k\Delta_f - \sum_{l=1}^{r} B_l\Delta_h d_l^M\right) + \left(\Gamma - W\Delta_g - \sum_{k=1}^{N} W_k\Delta_f - \sum_{l=1}^{r} B_l\Delta_h d_l^M\right)^{\mathrm{T}} P$$

is positive definite, where we have used Lemma 2.1 and the facts that $0 < \overline{\upsilon}_l \leq 1$ and

$$PB_l\Delta_h d_l^M + \left(PB_l\Delta_h d_l^M\right)^{\mathrm{T}} \leq PB_lH_l^{-1}B_l^TP/\overline{\upsilon}_l + 0.5\left(d_l^M\right)^2 \Delta_h H_l \Delta_h (1 + 1/\overline{\upsilon}_l).$$

According to Lemma 4.2, we only need to prove that $\overline{F}(u)$ is norm coercive and locally univalent. We first prove that $\overline{F}(u)$ is locally univalent. For any $u = (u_1, \ldots, u_n) \in \Re^n$, without loss of generality, by some rearrangement of u_i, we can assume that $u_i > 0$ if $i = 1, \ldots, p$; $u_i < 0$ if $i = p+1, \ldots, m$; and $u_i = 0$ if $i = m+1, \ldots, n$ for some integers $p \leq m \leq n$. Moreover, if $y \in \Re^n$ is sufficiently close to $u \in \Re^n$, without loss of generality, we can also assume that $y_i > 0$ if $i = 1, \ldots, p$; $y_i < 0$ if $i = p+1, \ldots, m$; $y_i > 0$ if $i = m+1, \ldots, m_1$;

$y_i < 0$ if $i = m_1+1,\ldots,m_2$; and $y_i = 0$ if $i = m_2+1,\ldots,n$ for some integers $m \le m_1 \le m_2 \le n$. It can be seen that

$$\left(u_i^+ - y_i^+\right)\left(u_i^- - y_i^-\right) = 0, \qquad i = 1,\ldots,n \tag{56}$$

$$\overline{F}(u) - \overline{F}(y) = \left[\overline{\Gamma} - WK_g - \sum_{k=1}^{N} W_k K_f - \sum_{l=1}^{r} B_l \tilde{d}_l K_h\right] \times (u^+ - y^+) + (u^- - y^-) \tag{57}$$

where $\overline{\Gamma} = \mathrm{diag}(\overline{\gamma}_1,\ldots,\overline{\gamma}_n)$, $K_g = \mathrm{diag}(K_g^1,\ldots,K_g^n)$, $K_f = \mathrm{diag}(K_f^1,\ldots,K_f^n)$, $K_h = \mathrm{diag}(K_h^1,\ldots,K_h^n)$, $0 < \tilde{d}_l \le d_l^M$

$$\overline{\gamma}_i = \begin{cases} \frac{c_i(u_i^+) - c_i(y_i^+)}{x_i^+ - y_i^+}, & x_i^+ \ne y_i^+ \\ \gamma_i, & \text{otherwise} \end{cases}$$
$$K_g^i = \begin{cases} \frac{g_i(u_i^+) - g_i(y_i^+)}{x_i^+ - y_i^+}, & x_i^+ \ne y_i^+ \\ \delta_i^g, & \text{otherwise} \end{cases}$$
$$K_f^i = \begin{cases} \frac{f_i(u_i^+) - f_i(y_i^+)}{x_i^+ - y_i^+}, & x_i^+ \ne y_i^+ \\ \delta_i^f, & \text{otherwise} \end{cases}$$
$$K_h^i = \begin{cases} \frac{h_i(u_i^+) - h_i(y_i^+)}{x_i^+ - y_i^+}, & x_i^+ \ne y_i^+ \\ \delta_i^h, & \text{otherwise.} \end{cases}$$

Then, $\overline{\Gamma}_i \ge \Gamma_i$, $K_g^i \le \delta_i^g$, $K_f^i \le \delta_i^f$, and $K_h^i \le \delta_i^h$ $(i = 1,\ldots,n)$.

If $\overline{F}(u) - \overline{F}(y) = 0$, then we have

$$u^- - y^- = -\left[\overline{\Gamma} - WK_g - \sum_{k=1}^{N} W_k K_f - \sum_{l=1}^{r} B_l \tilde{d}_l K_h\right](u^+ - y^+). \tag{58}$$

By (56), without loss of generality, we can assume that

$$u^+ - y^+ = \begin{bmatrix} z_1 \\ 0 \end{bmatrix} \quad u^- - y^- = \begin{bmatrix} 0 \\ z_2 \end{bmatrix}$$

where $z_1 \in \Re^k$ and $z_2 \in \Re^{n-k}$ for some integer k. Therefore, (58) can be written as

$$\begin{bmatrix} 0 \\ z_2 \end{bmatrix} = -\left[\overline{\Gamma} - WK_g - \sum_{k=1}^{N} W_k K_f - \sum_{l=1}^{r} B_l \tilde{d}_l K_h\right]\begin{bmatrix} z_1 \\ 0 \end{bmatrix}.$$

According to Lemma 4.3 and from $A^S > 0$, $\overline{\Gamma} - WK_g - \sum_{k=1}^{N} W_k K_f - \sum_{l=1}^{r} B_l \tilde{d}_l K_h$ is nonsingular. By matrix decomposition, we can easily obtain $z_1 = 0$ and $z_2 = 0$, respectively. Therefore, $u^+ = y^+$ and $u^- = y^-$, i.e., $u = y$, which implies that $\overline{F}(u)$ is locally univalent.

We next prove that $\overline{F}(u)$ is norm coercive. Suppose that there exists a sequence $\{u_m = (u_{m,1},\ldots,u_{m,n})^{\mathrm{T}}\}_{m=1}^{\infty}$ such that $\lim_{m\to\infty}\|u_m\| = \infty$. Then, there exists some index i such that $\lim_{m\to\infty}|c_i(u_{m,i}^+) + u_{m,i}^-| = \infty$, which implies that $\|\overline{F}(u_m)\| \to \infty$ in the case of the bounded activation functions $\tilde{g}(\cdot)$, $\tilde{f}(\cdot)$, and $\tilde{h}(\cdot)$. In the case of $\lim_{m\to\infty}\|\tilde{g}(u_m^+)\| = \infty$, $\lim_{m\to\infty}\|\tilde{f}(u_m^+)\| = \infty$, or $\lim_{m\to\infty}\|\tilde{h}(u_m^+)\| = \infty$, we will prove that $\|\overline{F}(u_m)\| \to \infty$.

By simple algebraic manipulations, it leads to

$$\begin{aligned}
&2\left[\tilde{g}^T(u_m^+) \quad \tilde{f}^T(u_m^+) \quad \tilde{h}^T(u_m^+)\right]\overline{P}\,\overline{F}(u_m) \\
&= 2\tilde{g}^T(u_m^+)\Delta_g^{-1}P\overline{F}(u_m) + 2\tilde{f}^T(u_m^+)\Delta_f^{-1}P\overline{F}(u_m) \\
&\quad + 2\tilde{h}^T(u_m^+)\Delta_h^{-1}P\overline{F}(u_m) \\
&\ge 2\tilde{g}^T(u_m^+)\Delta_g^{-1}P\Gamma\Delta_g^{-1}\tilde{g}(u_m^+) \\
&\quad - 2\tilde{g}^T(u_m^+)\Delta_g^{-1}PW\Delta_g\Delta_g^{-1}\tilde{g}(u_m^+) \\
&\quad - 2\tilde{g}^T(u_m^+)\Delta_g^{-1}P\sum_{k=1}^{N} W_k\Delta_f\Delta_f^{-1}\tilde{f}(u_m^+) \\
&\quad - 2\tilde{g}^T(u_m^+)\Delta_g^{-1}P\sum_{l=1}^{r} B_l\Delta_h\Delta_h^{-1}\tilde{h}(u_m^+) \\
&\quad + 2\tilde{g}^T(u_m^+)\Delta_g^{-1}Pu_m^- + 2\tilde{f}^T(u_m^+)\Delta_f^{-1}P\Gamma\Delta_f^{-1}\tilde{f}(u_m^+) \\
&\quad - 2\tilde{f}^T(u_m^+)\Delta_f^{-1}PW\Delta_g\Delta_g^{-1}\tilde{g}(u_m^+) \\
&\quad - 2\tilde{f}^T(u_m^+)\Delta_f^{-1}P\sum_{k=1}^{N} W_k\Delta_f\Delta_f^{-1}\tilde{f}(u_m^+) \\
&\quad - 2\tilde{f}^T(u_m^+)\Delta_f^{-1}P\sum_{l=1}^{r} B_l\Delta_h\Delta_h^{-1}\tilde{h}(u_m^+) \\
&\quad + 2\tilde{f}^T(u_m^+)\Delta_f^{-1}Pu_m^- + 2\tilde{h}^T(u_m^+)\Delta_h^{-1}P\Gamma\Delta_h^{-1}\tilde{h}(u_m^+) \\
&\quad - 2\tilde{h}^T(u_m^+)\Delta_h^{-1}PW\Delta_g\Delta_g^{-1}\tilde{g}(u_m^+) \\
&\quad - 2\tilde{h}^T(u_m^+)\Delta_h^{-1}P\sum_{k=1}^{N} W_k\Delta_f\Delta_f^{-1}\tilde{f}(u_m^+) \\
&\quad - 2\tilde{h}^T(u_m^+)\Delta_h^{-1}P\sum_{l=1}^{r} B_l\Delta_h\Delta_h^{-1}\tilde{h}(u_m^+) \\
&\quad + 2\tilde{h}^T(u_m^+)\Delta_h^{-1}Pu_m^-.
\end{aligned} \tag{59}$$

Let $\overline{P} = [(\Delta_g^{-1}P)^{\mathrm{T}} \ (\Delta_f^{-1}P)^{\mathrm{T}} \ (\Delta_h^{-1}P)^{\mathrm{T}}]^{\mathrm{T}}$, $\Delta_g^{-1}\tilde{g}(u_m^+) = \overline{g}(u_m^+)$, $\Delta_f^{-1}\tilde{f}(u_m^+) = \overline{f}(u_m^+)$, $\Delta_h^{-1}\tilde{h}(u_m^+) = \overline{h}(u_m^+)$, and by Lemma 2.1, (59) becomes

$$\begin{aligned}
&2\left[\tilde{g}^T(u_m^+) \quad \tilde{f}^T(u_m^+) \quad \tilde{h}^T(u_m^+)\right]\overline{P}\,\overline{F}(u_m) \\
&\ge \overline{g}^T(u_m^+)\Bigg[2P\Gamma - PWQ^{-1}W^TP - \Delta_gQ\Delta_g \\
&\qquad - \sum_{k=1}^{N}\left(PW_kQ_k^{-1}W_k^TP/\eta_k + \Delta_fQ_k\Delta_f\right) \\
&\qquad - \sum_{l=1}^{r}\Big(PB_lH_l^{-1}B_l^TP/\overline{\upsilon}_l \\
&\qquad\qquad + \left(d_l^M\right)^2\Delta_hH_l\Delta_h\Big)\Bigg]\overline{g}(u_m^+) \\
&\quad + \sum_{k=1}^{N}\left[\overline{g}^T(u_m^+)\Delta_fQ_k\Delta_f\overline{g}(u_m^+) - \overline{f}^T(u_m^+)\Delta_fQ_k\Delta_f\overline{f}(u_m^+)\right] \\
&\quad + \sum_{l=1}^{r}\left[\overline{g}^T(u_m^+)\left(d_l^M\right)^2\Delta_hH_l\Delta_h\overline{g}(u_m^+) - \overline{h}^T(u_m^+)\Delta_hH_l\Delta_h\overline{h}(u_m^+)\right] \\
&\quad + 2\overline{f}^T(u_m^+)P\Gamma\overline{f}(u_m^+) - 2\overline{f}^T(u_m^+)PW\Delta_g\overline{g}(u_m^+) \\
&\quad - 2\overline{f}^T(u_m^+)P\sum_{k=1}^{N}W_k\Delta_f\overline{f}(u_m^+)
\end{aligned}$$

$$\begin{aligned}&-2\overline{f}^T\left(u_m^+\right)P\sum_{l=1}^{r}B_l\Delta_h\overline{h}\left(u_m^+\right)\\&+2\overline{h}^T\left(u_m^+\right)P\Gamma\overline{h}\left(u_m^+\right)-2\overline{h}^T\left(u_m^+\right)PW\Delta_g\overline{g}\left(u_m^+\right)\\&-2\overline{h}^T\left(u_m^+\right)P\sum_{k=1}^{N}W_k\Delta_f\overline{f}\left(u_m^+\right)\\&-2\overline{h}^T\left(u_m^+\right)P\sum_{l=1}^{r}B_l\Delta_h\overline{h}\left(u_m^+\right)\\&\geq\lambda_m(\Xi)g_{fh}^T\left(u_m^+\right)g_{fh}\left(u_m^+\right)\end{aligned}\tag{60}$$

where $g_{fh}^T(u_m^+)=[\overline{g}^T(u^+)\quad \overline{f}^T(u^+)\quad \overline{h}^T(u^+)]$. Therefore, $\|\overline{F}(u_m)\|\geq\lambda_m(\Xi)\|\overline{P}\|\|g_{fh}(u_m^+)\|\to\infty$, which implies that $\overline{F}(u)$ is norm coercive. In combination with Lemma 4.2, (55) ensures the existence and uniqueness of the nonnegative equilibrium point of system (3).

Second, we will prove that (55) is also a sufficient condition to guarantee the global asymptotical stability of the nonnegative equilibrium point of system (3). It suffices to show that $\Theta_1>0$ ensures the global asymptotic stability of nonnegative equilibrium point.

Let $u^*=[u_1^*,\ldots,u_n^*]^{\mathrm{T}}$ be the nonnegative equilibrium point of system (3) and $x(t)=u(t)-u^*$. Then, model (3) is transformed into the following form:

$$\dot{x}(t)=-A(x(t))\left[C(x(t))-\sum_{k=1}^{N}W_kf(x(t-\overline{\tau}_k(t)))\right.\left.-Wg(x(t))-\sum_{l=1}^{r}B_l\int_{t-d_l(t)}^{t}h(x(s))\,\mathrm{d}s+J\right]\tag{61}$$

with $J=(J_1,\ldots,J_n)^{\mathrm{T}}$

$$J_i=\begin{cases}J_i^s, & u_i^*=0\\0, & u_i^*>0\end{cases}$$

where $J_i^s=c_i(u_i^*)-\sum_{j=1}^{n}w_{ij}\tilde{g}_j(u_j^*)-\sum_{k=1}^{N}\sum_{j=1}^{n}w_{ij}^k\tilde{f}_j(u_j^*)-\sum_{l=1}^{r}\sum_{j=1}^{n}b_{ij}^ld_l(t)\tilde{h}_j(u_j^*)+U_i$, the initial condition satisfies (52), and others are the same as those defined in (10).

Since u^* is the nonnegative equilibrium point of system (3), then, from (54), we know that $J_i\geq 0$ holds for all $i=1,\ldots,n$. This implies that $g_i(x_i(t))J_i\geq 0$, $f_i(x_i(t))J_i\geq 0$, and $h_i(x_i(t))J_i\geq 0$ hold for all $i=1,\ldots,n$ and $t\geq 0$.

Consider the same Lyapunov functional used in the proof of Theorem 2.3 with $\overline{d}_i=0$, and following the same routine as the proof of Theorem 2.3, we can show that $\Xi>0$ guarantees the global asymptotic stability of the nonnegative equilibrium point of system (3). Details are omitted. ■

Remark 4.1: If $g(\cdot)=f(\cdot)=h(\cdot)$, $d_l(t)=d_l$ is a constant delay in system (61), $l=1,\ldots,r$, and under the same assumptions of Theorem 4.1, we can prove that conditions (42) in Theorem 2.5 and (43) in Corollary 2.1 ensure the global asymptotic stability of the nonnegative equilibrium point of system (61).

Remark 4.2: In a similar routine to the proof of Theorem 3.1, we can easily derive the robust stability criterion from Theorem 4.1 in the case of uncertainties satisfying Assumption 3.1.

V. Illustrative Examples

In this section, we will use two examples to show the effectiveness of the obtained results.

Example 5.1: Consider the following Cohen–Grossberg neural networks with two neurons:

$$\begin{aligned}\dot{u}_1(t)=&\,a_1(u_1(t))[-u_1(t)-0.5g_1(u_1(t))\\&+g_2(u_2(t))-0.78g_1(u_1(t-\tau_{11}))\\&+0.2g_2(u_2(t-\tau_{12}))+1]\\\dot{u}_2(t)=&\,a_2(u_2(t))[-u_2(t)+0.1g_1(u_1(t))\\&-g_2(u_2(t))+0.9g_1(u_1(t-\tau_{21}))\\&-0.2g_2(u_2(t-\tau_{22}))+2]\end{aligned}\tag{62}$$

where $g_i(u_i(t))=0.5(|u_i(t)+1|-|u_i(t)-1|)$ and τ_{ij} are any bounded constant delays and $a_i(u_i(t))$ are some kinds of amplification functions satisfying Assumption 2.3, whose exact values of lower and upper bounds may be unknown $(i,j=1,2)$. Obviously, the results in [10] and [19] cannot be applied to this example. Pertaining to this example, the result in [27] and [7, Th. 2] are not satisfied. Applying Corollary 2.3 of this paper, we have $D=\mathrm{diag}(6.1842,8.0659)$, $Q_1=\mathrm{diag}(4.7152,3.6839)$, and $Q_2=\mathrm{diag}(4.2247,3.4643)$. Therefore, the neural network (62) is globally asymptotically stable. Moreover, if we let $a_i(u_i(t))\equiv 1$ in (62), $i=1,2$, [24, Ths. 1 and 2], [47, Th. 1], and [4, Th. 3] are not satisfied either.

Example 5.2: Consider the following two Cohen–Grossberg neural networks with two neuron:

$$\begin{aligned}\dot{y}_1(t)=&\,y_1(t)[-y_1(t)+w_{11}g_1(y_1(t))\\&+w_{12}g_2(y_2(t))+w_{11}^1g_1(y_1(t-2))\\&+w_{12}^1g_2(y_2(t-3))+U_1]\\\dot{y}_2(t)=&\,y_2(t)[-y_2(t)+w_{21}g_1(y_1(t))\\&+w_{22}g_2(y_2(t))+w_{21}^1g_1(y_1(t-2))\\&+w_{22}^1g_2(y_2(t-3))+U_2]\end{aligned}\tag{63}$$

$$\begin{aligned}\dot{u}_1(t)=&\,a_1(u_1(t))[-u_1(t)+w_{11}g_1(u_1(t))\\&+w_{12}g_2(u_2(t))+w_{11}^1g_1(u_1(t-2))\\&+w_{12}^1g_2(u_2(t-3))+U_1]\\\dot{u}_2(t)=&\,a_2(u_2(t))[-u_2(t)+w_{21}g_1(u_1(t))\\&+w_{22}g_2(u_2(t))+w_{21}^1g_1(u_1(t-2))\\&+w_{22}^1g_2(u_2(t-3))+U_2]\end{aligned}\tag{64}$$

where $a_i(u_i)=1/(1+|u_i|)$, $g_i(u_i)=0.5[u_i+\tanh(u_i)]$, $W=\begin{bmatrix}-2&1\\-1&-1\end{bmatrix}$, and $W_1=\begin{bmatrix}-1&1\\1&1\end{bmatrix}$.

Pertaining to this example, [28, Th. 4] cannot judge the stability of system (63). Applying Corollary 2.2 and Theorem 2.5, we can show that systems (63) and (64) are both globally asymptotically stable, respectively.

In the following, we set initial conditions $y(s)=(2,1)^{\mathrm{T}}$ and $u(s)=(2,1)^{\mathrm{T}}$ for $s\in[-3,0]$, respectively.

When $U=(1,-0.2)^{\mathrm{T}}$, the equilibrium points of system (63) are $(0,0)^{\mathrm{T}}$, $(0,-0.2)^{\mathrm{T}}$, $(0.2520,0)^{\mathrm{T}}$, and $(0.1511,-0.2)^{\mathrm{T}}$. Among them, $(0.2520,0)^{\mathrm{T}}$ is the nonnegative equilibrium point. $(0.1511,-0.2)^{\mathrm{T}}$ is the unique equilibrium point of system (64). Obviously, different dynamics between systems (63) and (64) are caused by different amplification functions.

VI. Conclusion

Under different assumptions of amplification function, some sufficient criteria are derived for the global asymptotic stability of a class of Cohen–Grossberg neural networks with both multiple time-varying delays and continuously distributed delays, which are independent of the size of the time-varying delays and amplification functions. The results can be applied to some special cases of Cohen–Grossberg neural networks, for example, Hopfield neural networks and Lotka–Volterra recurrent neural networks. Two numerical examples are employed to demonstrate the effectiveness of the obtained results.

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