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Two new operators in rough set theory with applications to fuzzy sets

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Abstract

In this paper, two new operators are introduced for the rough set theory. Using them, two inequalities well known in the rough set theory can now be modified to become equalities. With this change, no information will be lost in the new expressions. Hence, many properties in rough set theory can be improved and in particular, the union, the intersection, and the complement operations can be redefined based on the two equalities. Furthermore, the collection of rough sets of an approximation space forms a Boolean algebra under these new operators. Finally, roughness properties of fuzzy sets are analyzed using the new operations.

Keywords: Roughness of fuzzy sets; Rough set theory; Certain increment operator; Uncertain decrement operator; Boolean algebra

1. Introduction

In 1982, Pawlak introduced the rough set theory [7], which has emerged as another major mathematical tool for modelling the vagueness present in human

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classification mechanism. This concept is fundamental to the examination of granularity in knowledge [9,20,21]. It is a concept that has many applications in machine learning, pattern recognition, decision support systems, expert systems, data analysis, and data mining, among others. The theory of fuzzy sets introduced by Zadeh has provided a useful mathematical tool for describing the behavior of systems that are too complex or ill-defined to admit precise mathematical analysis by classical methods and tools. Extensive applications of the fuzzy set theory have been found in various fields. There have been many papers studying the connections and differences of fuzzy set theory and rough set theory [1–3,5,10–12]. One important example of such studies is the roughness of fuzzy sets where some notions of rough sets and fuzzy sets are integrated [1].

The R-lower and R-upper approximations of X are two basic concepts in rough set theory [14,15,19]. They have many properties that have become the foundation of rough set theory [13]. In particular the properties $\underline{R}(X \cup Y) \supseteq \underline{R}X \cup \underline{R}Y$ and $\overline{R}(X \cap Y) \subseteq \overline{R}X \cap \overline{R}Y$ are of great importance, for they do not allow for step by step computation of approximations. In other words, approximations of $X \cup Y$ cannot be done in general by the ones of X and of Y. These properties are a logical, formal consequence of the assumed definition of the knowledge base expressed in the form of approximations [8]. They have brought inconvenience and difficulties in many fields. The purpose of this paper is to show that these inequalities can be modified to become equalities by defining two new operators and use the two equalities to establish results for the algebraic operation of rough sets, the roughness of rough sets, and the roughness of fuzzy sets.

The paper is organized as follows. In Section 2, we discuss the basics of rough sets and provide some definitions. In Section 3, we define two new operators, the certain increment operator and the uncertain decrement operator, and we establish their properties. In Section 4, we define the operations in rough sets and we discuss the properties of these operations. In Section 5, we introduce the Boolean algebra of rough sets and we show that the collection of rough sets of an approximation space forms a Boolean algebra under the present new operations. In Section 6, we establish results for the roughness of fuzzy sets. In Section 7, we conclude the present paper.

2. Definitions and notation

We introduce in this section some definitions and notation used in the present paper.

Definition 2.1. Let U be a finite and non-empty set which is called the universe. Let R be an equivalence relation on U. We use U/R to denote the family of all equivalence classes of R (or classifications of U), and we use $[x]_R$ to denote an equivalence class in R containing an element $x \in U$.

We immediately have the following result.

Theorem 2.1. *Let U be the universe and let R be an equivalence relation. The equivalence class $[y]_R = [x]_R$ if $y \in [x]_R$.*

Proof. For any $z \in [y]_R, zRy$. If $y \in [x]_R$, then yRx. From the transitivity of equivalence relations, we have zRx. Thus, $z \in [x]_R$ and $[y]_R \subseteq [x]_R$. Similarly, for any $z \in [x]_R, zRx$. If $y \in [x]_R$, then xRy. From the transitivity of equivalence relations, we have zRy. Thus, $z \in [y]_R$ and $[x]_R \subseteq [y]_R$. Therefore, $[x]_R = [y]_R$. $\square$

It is also concluded that for any $x, y \in U$, either $[x]_R = [y]_R$ or $[x]_R \cap [y]_R = \emptyset$.

Definition 2.2. Let U be the universe and let R be an equivalence relation on U. For any subset $X \subseteq U$, the pair $S = (U, R)$ is called an approximation space. The two subsets

$$\underline{R}X = \{x \in U \mid [x]_R \subseteq X\}, \tag{1}$$

$$\overline{R}X = \{x \in U \mid [x]_R \cap X \neq \emptyset\} \tag{2}$$

are called the R-lower and R-upper approximation of X, respectively. $R(X) = (\underline{R}X, \overline{R}X)$ is called the rough set of X in S. The rough set $R(X)$ denotes the description of X under the present knowledge, i.e., the classification of U.

We will use $POS_R(X) = \underline{R}X$ to denote R-positive region of X; $NEG_R(X) = U - \overline{R}X$ to denote R-negative region of X; and $BN_R(X) = \overline{R}X - \underline{R}X$ to denote the R-borderline region of X. The positive region $POS_R(X)$ or $\underline{R}X$ is the collection of those objects which can be classified with full certainty as members of the set X, using knowledge R. The negative region $NEG_R(X)$ is the collection of objects which can be determined without any ambiguity, employing knowledge R, that they do not belong to the set X; that is, they belong to the complement $\neg X$ of X. The borderline region is, in a sense, the undecidable area of the universe, i.e., none of the objects belonging to the borderline region can be classified with certainty into X or its complement $\neg X$ as far as knowledge R is concerned.

Definition 2.3. Let U be the universe and let R be an equivalence relation on U. Let $X \subseteq U$. The set $W = \{R(X) \mid X \subseteq U\}$ is called the set of rough sets on U.

Remark 2.1. For any set $X \subseteq U$, there is only one corresponding rough set $R(X)$ in W.

Definition 2.4. Let U be the universe and let R be an equivalence relation on U. Let $X, Y \subseteq U$. X is called R-included in Y (i.e., $RX \subseteq RY$) if $\underline{R}X \subseteq \underline{R}Y$ and $\overline{R}X \subseteq \overline{R}Y$. X and Y are said to be R-equal (i.e., $RX = RY$) if $\underline{R}X = \underline{R}Y$ and $\overline{R}X = \overline{R}Y$.

Definition 2.5. Let U be the universe and let R be an equivalence relation on U. For $X \subseteq U$, a measure of roughness of the set X is defined as

$$\rho_X = 1 - |\underline{R}X|/|\overline{R}X|, \tag{3}$$

where $|\cdot|$ denotes the cardinality of a set. The roughness of set X is reflected by the ratio of the number of objects in its lower approximation to that in its upper approximation—the greater the value of the ratio, the lower the roughness.

Remark 2.2
(i) Since $\underline{R}X \subseteq X \subseteq \overline{R}X$, we have $0 \leqslant \rho_X \leqslant 1$;
(ii) By convention, when $X = \emptyset$, $\underline{R}X = \emptyset = \overline{R}X$ and $|\underline{R}X|/|\overline{R}X| = 1$, i.e., $\rho_X = 0$; and
(iii) $\rho_X = 0$ if and only if X is an exact set, i.e., $\underline{R}X = X = \overline{R}X$.

The above definitions can be found in [8] and are required in the present paper. We next introduce two new definitions.

Definition 2.6. Let U be the universe and let R be an equivalence relation on U. Let $X \subseteq U$. For any element $x \in X$, the two sets

$$h_X(x) = [x]_R - X, \tag{4}$$

$$l_X(x) = [x]_R - h_X(x) \tag{5}$$

are called the basic factor of inducing rough and the correlation basic factor of inducing rough of X, respectively. We can see clearly that $l_X(x) \cap h_X(x) = \emptyset$, $l_X(x) \cup h_X(x) = [x]_R$. $h_X(x)$ is the collection of those objects which are in $[x]_R$ but not in the set X. $l_X(x)$ is the collection of those objects which are in both $[x]_R$ and X.

Remark 2.3. Obviously for any $x \in X$, $h_X(x) = \emptyset$, if $x \in \underline{R}X$, i.e., the basic factor of inducing rough is the empty set, and the correlation basic factor of inducing rough is the equivalence class containing the element x. Thus, $[x]_R = l_X(x)$ is called R-certain information of X. Otherwise, $h_X(x) \neq \emptyset$ indicates that the equivalence class $[x]_R$ contains some elements which induce rough of the set X. So, in this case, $[x]_R$ is called R-uncertain information of X. The collection of R-certain information of X is the R-positive region of X, and the collection of R-uncertain information of X is the R-borderline region of X.

Definition 2.7. Let U be the universe and let R be an equivalence relation on U. Let $X \subseteq U$. For any element $x \in X$, the two sets

$$H(X) = \bigcup\{h_X(x) \mid x \in BN_R(X) \cap X\}, \tag{6}$$

$$L(X) = \bigcup\{l_X(x) \mid x \in BN_R(x) \cap X\} \tag{7}$$

are called R-inducing rough region and R-inducing rough correlation region of X, respectively.

Remark 2.4. Obviously, $BN_R(X) = H(X) \cup L(X)$ and $H(X) \cap L(X) = \emptyset$. That is to say, the R-borderline region of X is divided into two parts: the R-inducing rough region and R-inducing rough correlation region of X. This division will help us to analyze and discuss properties of the operations of rough sets.

3. Certain increment operator and uncertain decrement operator

The following two theorems are taken from [8]. They serve as the starting point for our analysis in the present paper.

Theorem 3.1. *Let U be the universe and let R be an equivalence relation on U. Let $X, Y \subseteq U$. We have*

$$\overline{R}(X \cup Y) = \overline{R}X \cup \overline{R}Y, \tag{8}$$

$$\underline{R}(X \cap Y) = \underline{R}X \cap \underline{R}Y. \tag{9}$$

Theorem 3.2. *Let U be the universe and let R be an equivalence relation on U. Let $X, Y \subseteq U$. We have*

$$\underline{R}(X \cup Y) \supseteq \underline{R}X \cup \underline{R}Y, \tag{10}$$

$$\overline{R}(X \cap Y) \subseteq \overline{R}X \cap \overline{R}Y. \tag{11}$$

From the above two theorems, we can draw the following conclusions.

(1) The certain information of $X \cup Y$ may be more than the union of the certain information of X and Y.
(2) The uncertain information of $X \cap Y$ may be less than the intersection of the uncertain information of X and Y.

Due to these reasons, the operations of union, intersection, and complement in rough set theory cannot be quantitatively analyzed. We will solve this problem in the present paper by introducing two new operators—the certain increment operator and the uncertain decrement operator.

Definition 3.1 (*The certain increment operator*). Let U be the universe and let R be an equivalence relation on U. Let $X, Y \subseteq U$. When X is extended by Y (i.e., $X \cup Y$), $\underline{Z}_{(\cdot)}(\cdot) : U \times U \to U$ defined by

$$\underline{Z}_X(Y) = \bigcup\{[x]_R \,|\, x \in L(X), l_X(x) \nsubseteq Y, h_X(x) \subseteq Y\} \tag{12}$$

is called the certain increment operator of X.

$\underline{Z}_X(Y)$ is just the collection of those objects in which the certain information of $X \cup Y$ is larger than the union of the certain information of X and Y.

The certain increment operator has the following properties.

Property 3.1

$$\underline{Z}_X(Y) = \underline{Z}_Y(X). \tag{13}$$

Proof. For any $[x]_R \subseteq \underline{Z}_X(Y)$ we have $h_X(x) \subseteq Y$, i.e., for any $y \in h_X(x)$, $y \in Y$. Obviously, $y \in h_X(x) \subseteq [x]_R$, hence $[x]_R = [y]_R$, $y \in [y]_R$. From (5), $y \in l_Y(y)$. So $h_X(x) \subseteq l_Y(y)$, and $l_X(x) \supseteq h_Y(y)$, which yields $h_Y(y) \subseteq X$. Since $y \in h_X(x)$, so $y \notin X$, and $y \in l_Y(y)$. Therefore $l_Y(y) \nsubseteq X$. From (12), $[x]_R = [y]_R \subseteq \underline{Z}_Y(X)$, which yields $\underline{Z}_X(Y) \subseteq \underline{Z}_Y(X)$. Similarly, we have $\underline{Z}_Y(X) \subseteq \underline{Z}_X(Y)$. Therefore, $\underline{Z}_X(Y) = \underline{Z}_Y(X)$. $\square$

Property 3.2

$$\underline{Z}_X(\emptyset) = \emptyset, \tag{14}$$

$$\underline{Z}_X(X) = \emptyset, \tag{15}$$

$$\underline{Z}_X(\neg X) = BN_R(X) = \overline{R}X - \underline{R}X. \tag{16}$$

Proof. According to Definition 3.1, we can easily get $\underline{Z}_X(\emptyset) = \emptyset$ and $\underline{Z}_X(X) = \emptyset$. Next, we will prove (16). For any $x \in BN_R(X)$, we have $l_X(x) \nsubseteq (\neg X)$ and $h_X(x) \subseteq (\neg X)$. Hence, $x \in \underline{Z}_X(\neg X)$. Thus, $BN_R(X) \subseteq \underline{Z}_X(\neg X)$. From Definition 3.1, obviously, $\underline{Z}_X(\neg X) \subseteq BN_R(X)$. Therefore, $\underline{Z}_X(\neg X) = BN_R(X) = \overline{R}X - \underline{R}X$. $\square$

Property 3.3

$$\underline{Z}_X(Y) \cap (\underline{R}(X) \cup \underline{R}(Y)) = \emptyset. \tag{17}$$

Proof. (i) For any $x \in \underline{Z}_X(Y)$, $x \in L(X)$. Hence, $h_X(x) \neq \emptyset$, i.e., $[x]_R \not\subseteq X$. Thus, $x \notin \underline{R}(X)$, that is to say, $\underline{Z}_X(Y) \cap \underline{R}(X) = \emptyset$. (ii) For any $x \in \underline{Z}_X(Y)$, $l_X(x) \not\subseteq Y$. Hence, $[x]_R \not\subseteq Y$, i.e., $\underline{Z}_X(Y) \cap \underline{R}(Y) = \emptyset$. From (i) and (ii), we have $\underline{Z}_X(Y) \cap (\underline{R}(X) \cup \underline{R}(Y)) = \emptyset$. □

Definition 3.2 (*The uncertain decrement operator*). Let U be the universe and let R be an equivalence relation on U. Let $X, Y \subseteq U$. When X is cut by Y (i.e., $X \cap Y$), $\overline{Z}_{(\cdot)}(\cdot) : U \times U \to U$ defined by

$$\overline{Z}_X(Y) = \bigcup \{[x]_R \,|\, x \in L(X), l_X(x) \cap Y = \emptyset, h_X(x) \cap Y \neq \emptyset\}, \tag{18}$$

is called the uncertain decrement operator of X.

$\overline{Z}_{(X)}(Y)$ is just the collection of those objects in which the uncertain information of $X \cap Y$ is less than the intersection of the uncertain information of X and Y.

The uncertain decrement operator has the following properties.

Property 3.4

$$\overline{Z}_X(Y) = \overline{Z}_Y(X). \tag{19}$$

Proof. For any $[x]_R \subseteq \overline{Z}_X(Y)$, $h_X(x) \cap Y \neq \emptyset$. Hence, there exists a $y \in h_X(x) \cap Y$. Obviously, $y \in [x]_R$, $[x]_R = [y]_R$. From (5), we obtain $l_Y(y) = [y]_R \cap Y = [x]_R \cap Y$, and for $l_X(x) \cap Y = \emptyset$, $l_Y(y) = h_X(x) \cap Y$. Hence, $l_Y(y) \cap X = \emptyset$, $l_Y(y) \subseteq h_X(x)$, and for $[x]_R = [y]_R$, $h_Y(y) \supseteq l_X(x)$. Thus, $h_Y(y) \cap X \neq \emptyset$. From (18), $[x]_R = [y]_R \subseteq \overline{Z}_Y(X)$, which yields $\overline{Z}_X(Y) \subseteq \overline{Z}_Y(X)$. Similarly, we have $\overline{Z}_Y(X) \subseteq \overline{Z}_X(Y)$. Therefore, $\overline{Z}_X(Y) = \overline{Z}_Y(X)$. □

Property 3.5

$$\overline{Z}_X(\emptyset) = \emptyset, \tag{20}$$

$$\overline{Z}_X(X) = \emptyset, \tag{21}$$

$$\overline{Z}_X(\neg X) = BN_R(X). \tag{22}$$

Proof. According to Definition 3.2, we can easily get $\overline{Z}_X(\emptyset) = \emptyset$ and $\overline{Z}_X(X) = \emptyset$. Next, we will prove $\overline{Z}_X(\neg X) = BN_R(X)$. For any $x \in BN_R(X)$, $h_X(x) \neq \emptyset$. Hence, $h_X(x) \cap (\neg X) \neq \emptyset$. We also have $l_X(x) \cap (\neg X) = \emptyset$. Hence, $x \in \overline{Z}_X(\neg X)$, that is to say, $BN_R(X) \subseteq \overline{Z}_X(\neg X)$. From Definition 3.2, obviously, we have $\overline{Z}_X(\neg X) \subseteq BN_R(X)$. Therefore, $\overline{Z}_X(\neg X) = BN_R(X)$. □

Property 3.6

$$\overline{Z}_X(Y) \subseteq \overline{R}(X) \cap \overline{R}(Y). \tag{23}$$

Proof. (i) For any $x \in \overline{Z}_X(Y)$, we have $x \in L(X)$. Hence, $l_X(x) \neq \emptyset$, i.e., $[x]_R \cap X \neq \emptyset$. From Definition 2.2, $x \in \overline{R}(X)$. Thus, $\overline{Z}_X(Y) \subseteq \overline{R}(X)$. (ii) For any $x \in \overline{Z}_X(Y)$, we also have $h_X(x) \cap Y \neq \emptyset$, i.e., $[x]_R \cap Y \neq \emptyset$. From Definition 2.2, we have $x \in \overline{R}(Y)$. Thus, $\overline{Z}_X(Y) \subseteq \overline{R}(Y)$. From (i) and (ii), we have $\overline{Z}_X(Y) \subseteq \overline{R}(X) \cap \overline{R}(Y)$. □

Now we can modify Theorem 3.2 using the two new operators.

Theorem 3.3. *Let U be the universe and let R be an equivalence relation on U. Let $X, Y \subseteq U$. We have*

$$\underline{R}(X \cup Y) = \underline{R}X \cup \underline{R}Y \cup \underline{Z}_X(Y), \tag{24}$$

$$\overline{R}(X \cap Y) = \overline{R}X \cap \overline{R}Y - \overline{Z}_X(Y). \tag{25}$$

Proof. $\forall x \in \underline{R}(X \cup Y)$, $[x]_R \subseteq X \cup Y$, i.e. $[x]_R \subseteq X$ or $[x]_R \subseteq Y$ or ($l_X(x) \nsubseteq Y$ and $h_X(x) \subseteq Y$). Hence $[x]_R \subseteq \underline{R}X \cup \underline{R}Y \cup \underline{Z}_X(Y)$, which yields $\underline{R}(X \cup Y) \subseteq \underline{R}X \cup \underline{R}Y \cup \underline{Z}_X(Y)$. Similarly $\forall x \in \underline{R}X \cup \underline{R}Y \cup \underline{Z}_X(Y)$, $[x]_R \subseteq \underline{R}X \cup \underline{R}Y \cup \underline{Z}_X(Y)$. If $[x]_R \subseteq \underline{R}(x)$ or $[x]_R \subseteq \underline{R}(y)$, it is clear that $[x]_R \subseteq X \cup Y$; if $[x]_R \subseteq \underline{Z}_X(Y)$, we have $l_X(x) \subseteq X$, $h_X(x) \subseteq Y$. So $[x]_R = l_X(x) \cup h_X(x) \subseteq X \cup Y$. Hence $[x]_R \subseteq \underline{R}(X \cup Y)$, which yields $\underline{R}X \cup \underline{R}Y \cup \underline{Z}_X(Y) \subseteq \underline{R}(X \cup Y)$. Thus, $\underline{R}(X \cup Y) = \underline{R}X \cup \underline{R}Y \cup \underline{Z}_X(Y)$.

$\forall x \in \underline{R}(X \cap Y)$, $[x]_R \cap (X \cap Y) \neq \emptyset$ iff $[x]_R \cap X \neq \emptyset$ and $[x]_R \cap Y \neq \emptyset$ and at least exist a $y \in [x]_R$, $l_X(y) \cap Y \neq \emptyset$, that is to say, $[y]_R \nsubseteq \overline{Z}_X(Y)$. Since $y \in [x]_R$, $[x]_R = [y]_R$. Hence $[x]_R \nsubseteq \overline{Z}_X(Y)$. Thus $\overline{R}(X \cap Y) = \overline{R}X \cap \overline{R}Y - \overline{Z}_X(Y)$. □

In order to understand the two operators better, let us consider a simple example due to Pawlak in [8].

Example 3.1. Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$ and let R be an equivalence relation on U with the following equivalence classes:

$$E_1 = \{x_1, x_4, x_8\},$$

$$E_2 = \{x_2, x_5, x_7\},$$

$$E_3 = \{x_3\},$$

$$E_4 = \{x_6\},$$

namely, $U/R = \{E_1, E_2, E_3, E_4\}$.

Let $X_1 = \{x_1, x_4, x_7\}$ and $X_2 = \{x_2, x_8\}$. Then,

$$\underline{R}(X_1 \cup X_2) = E_1, \tag{26}$$

$$\underline{R}X_1 = \emptyset, \quad \underline{R}X_2 = \emptyset.$$

Hence

$$\underline{R}(X_1 \cup X_2) \supset \underline{R}X_1 \cup \underline{R}X_2. \tag{27}$$

According to Definitions 2.6 and 2.7, we can get

$$\overline{R}X_1 = E_1 \cup E_2, \quad BN_R(X_1) = E_1 \cup E_2,$$
$$L(X_1) = \{x_1, x_4, x_7\},$$
$$l_{X_1}(x_1) = \{x_1, x_4\}, \quad l_{X_1}(x_4) = \{x_1, x_4\}, \quad l_{X_1}(x_7) = \{x_7\},$$
$$h_{X_1}(x_1) = \{x_8\}, \quad h_{X_1}(x_4) = \{x_8\},$$
$$h_{X_1}(x_7) = \{x_2, x_5\}.$$

From Definition 3.1

$$\underline{Z}_{X_1}(X_2) = [x_1]_R \cup [x_4]_R = E_1. \tag{28}$$

Hence, from Theorem 3.3

$$\underline{R}(X_1 \cup X_2) = \underline{R}X_1 \cup \underline{R}X_2 \cup \underline{Z}_{X_1}(X_2) = E_1. \tag{29}$$

Remark 3.1. From Theorem 3.2 or (27), we have $\underline{R}X_1 \cup \underline{R}X_2 \subseteq \underline{R}(X_1 \cup X_2)$. That is to say, the information will be lost if the lower approximation $X_1 \cup X_2$ are only denoted by $\underline{R}X_1$ and $\underline{R}X_2$. As the former does not take into account the effects of the combination of X_1 and X_2, i.e., Eq. (28) indicates the information lost during the operation of (24). Thus, by means of the new operator $\underline{Z}_{X_1}(X_2)$ in Definition 3.1, the inequality (27) can be modified to become equality (29). With this change, no R-certain information of the set $X_1 \cup X_2$ will be lost.

Next, let $Y_1 = \{x_1, x_3, x_5\}$ and $Y_2 = \{x_2, x_3, x_4, x_6\}$. Then,

$$\overline{R}(Y_1 \cap Y_2) = E_3, \tag{30}$$
$$\overline{R}Y_1 = E_1 \cup E_2 \cup E_3, \quad \overline{R}Y_2 = E_1 \cup E_2 \cup E_3 \cup E_4 = U,$$
$$\overline{R}Y_1 \cap \overline{R}Y_2 = E_1 \cup E_2 \cup E_3.$$

Hence

$$\overline{R}(Y_1 \cap Y_2) \subset \overline{R}Y_1 \cap \overline{R}Y_2. \tag{31}$$

According to Definitions 2.6 and 2.7

$$\underline{R}Y_1 = E_3, \quad BN_R(Y_1) = E_1 \cup E_2,$$
$$L(Y_1) = \{x_1, x_5\},$$
$$l_{Y_1}(x_1) = \{x_1\}, \quad l_{Y_1}(x_5) = \{x_5\},$$
$$h_{Y_1}(x_1) = \{x_4, x_8\}, \quad h_{Y_1}(x_5) = \{x_2, x_7\}.$$

From Definition 3.2

$$\overline{Z}_{Y_1}(Y_2) = [x_1]_R \cup [x_5]_R = E_1 \cup E_2. \tag{32}$$

Hence, from Theorem 3.3

$$\overline{R}(Y_1 \cap Y_2) = \overline{R}Y_1 \cap \overline{R}Y_2, -\overline{Z}_{Y_1}(Y_2) = E_3. \tag{33}$$

Remark 3.2. From (31), we can see that the intersection of the uncertain information of Y_1 and Y_2 are larger than the uncertain information of $Y_1 \cap Y_2$, while (32) is just those uncertain information (i.e., $E_1 \cup E_2$). Thus, the inequality (31) can be replaced by the equality (33) using the new operator $\overline{Z}_{Y_1}(Y_2)$. That is to say, we can reduce the R-uncertain information of Y_1 and Y_2 in (31).

Comparing (26) with (29) and (30) with (33), we can find that there is no information lost when we computed the union and intersection of rough sets by using the new operators. They can retain the reality and integrity of all the information.

4. The operations in rough sets

In the previous section, certain increment operator and uncertain decrement operator have been defined. In this section, we will define the operations in rough sets.

Definition 4.1. Let U be the universe and let R be an equivalence relation on U. Let $X, Y \subseteq U$. The union, the intersection, the complement, and the difference set of rough sets are defined as follows:

$$RX \cup RY = R(X \cup Y) = (\underline{R}X \cup \underline{R}Y \cup \underline{Z}_X(Y), \overline{R}X \cup \overline{R}Y), \tag{34}$$

$$RX \cap RY = R(X \cap Y) = (\underline{R}X \cap \underline{R}Y, \overline{R}X \cap \overline{R}Y - \overline{Z}_X(Y)), \tag{35}$$

$$R(\neg X) = \neg RX = (U - \overline{R}X, U - \underline{R}X), \tag{36}$$

$$RX - RY = R(X) \cap R(\neg Y) = (\underline{R}X - \overline{R}Y, \overline{R}X - \underline{R}Y - \overline{Z}_X(\neg Y)). \tag{37}$$

According to Definition 4.1, we can get many useful properties of these operations. Let A, B, C be non-empty subsets of U and let R be an equivalence relation on U. We have $R(A), R(B), R(C) \in W$. For convenience, let $\underline{A} = \underline{R}A$, $\overline{A} = \overline{R}A$. We now enumerate and prove some properties of the operations in rough sets.

(1) Commutative law

$$R(A) \cup R(B) = R(B) \cup R(A), \tag{38}$$
$$R(A) \cap R(B) = R(B) \cap R(A). \tag{39}$$

Proof

$$\begin{aligned}
R(A) \cup R(B) &= (\underline{A} \cup \underline{B} \cup \underline{Z}_A(B), \overline{A} \cup \overline{B}) = (\underline{B} \cup \underline{A} \cup \underline{Z}_A(B), \overline{B} \cup \overline{A}) \\
&= (\underline{B} \cup \underline{A} \cup \underline{Z}_B(A), \overline{B} \cup \overline{A}) = R(B) \cup R(A), \\
R(A) \cap R(B) &= (\underline{A} \cap \underline{B}, \overline{A} \cap \overline{B} - \overline{Z}_A(B)) = (\underline{B} \cap \underline{A}, \overline{B} \cap \overline{A} - \overline{Z}_A(B)) \\
&= (\underline{B} \cap \underline{A}, \overline{B} \cap \overline{A} - \overline{Z}_B(A)) = R(B) \cap R(A). \qquad \square
\end{aligned}$$

(2) Associative law

$$(R(A) \cup R(B)) \cup R(C) = R(A) \cup (R(B) \cup R(C)). \tag{40}$$
$$(R(A) \cap R(B)) \cap R(C) = R(A) \cap (R(B) \cap R(C)). \tag{41}$$

Proof

$$\begin{aligned}
(R(A) \cup R(B)) \cup R(C) &= ((\underline{A \cup B}) \cup \underline{C} \cup \underline{Z}_{A\cup B}(C), \overline{A \cup B} \cup \overline{C}) \\
&= (\underline{(A \cup B) \cup C}, \overline{(A \cup B) \cup C}) \\
&= (\underline{A \cup (B \cup C)}, \overline{A \cup (B \cup C)}) \\
&= (\underline{A} \cup \underline{(B \cup C)} \cup \underline{Z}_A(B \cup C), \overline{A} \cup \overline{(B \cup C)}) \\
&= R(A) \cup (R(B) \cup R(C)) \\
(R(A) \cap R(B)) \cap R(C) &= (\underline{A \cap B} \cap \underline{C}, \overline{A \cap B} \cap \overline{C} - \overline{Z}_{A\cap B}(C)) \\
&= (\underline{(A \cap B) \cap C}, \overline{(A \cap B) \cap C}) \\
&= (\underline{A \cap (B \cap C)}, \overline{A \cap (B \cap C)}) \\
&= (\underline{A} \cap \underline{B \cap C}, \overline{A} \cap \overline{B \cap C} - \overline{Z}_A(B \cap C)) \\
&= R(A) \cap (R(B) \cap R(C)). \qquad \square
\end{aligned}$$

(3) Distributive law

$$R(A) \cup (R(B) \cap R(C)) = (R(A) \cup R(B)) \cap (R(A) \cup R(C)), \tag{42}$$
$$R(A) \cap (R(B) \cup R(C)) = (R(A) \cap R(B)) \cup (R(A) \cap R(C)). \tag{43}$$

Proof

$$\begin{aligned}
R(A)\cup(R(B)\cap R(C)) &= (\underline{A}\cup\underline{B\cap C}\cup\underline{Z}_A(B\cap C),\overline{A}\cup\overline{B\cap C})\\
&= (\underline{A\cup(B\cap C)},\overline{A\cup(B\cap C)})\\
&= (\underline{(A\cup B)\cap(A\cup C)},\overline{(A\cup B)\cap(A\cup C)})\\
&= (\underline{A\cup B}\cap\underline{A\cup C},\overline{A\cup B}\cap\overline{A\cup C}-\overline{Z}_{A\cup B}(A\cup C)\\
&= (R(A)\cup R(B))\cap(R(A)\cup R(C)),
\end{aligned}$$

$$\begin{aligned}
(R(A)\cap R(B))\cup(R(A)\cap R(C)) &= (\underline{A\cap B}\cup\underline{A\cap C}\cup\underline{Z}_{A\cap B}(A\cap C),\overline{A\cap B}\\
&\quad\cup\overline{A\cap C})\\
&= (\underline{(A\cap B)\cup(A\cap C)},\overline{(A\cap B)\cup(A\cap C)})\\
&= (\underline{A\cap(B\cup C)},\overline{A\cap(B\cup C)})\\
&= (\underline{A}\cap\underline{B\cup C},\overline{A}\cap\overline{B\cup C}-\overline{Z}_A(B\cup C))\\
&= R(A)\cap(R(B)\cup R(C)). \qquad\square
\end{aligned}$$

(4) Idempotent law

$$R(A)\cup R(A)=R(A), \tag{44}$$

$$R(A)\cap R(A)=R(A). \tag{45}$$

Proof

$$R(A)\cup R(A)=(\underline{A}\cup\underline{A}\cup\underline{Z}_A(A),\overline{A}\cup\overline{A})=(\underline{A},\overline{A})=R(A),$$

$$R(A)\cap R(A)=(\underline{A}\cap\underline{A},\overline{A}\cap\overline{A}-\overline{Z}_A(A))=(\underline{A},\overline{A})=R(A). \qquad\square$$

(5) 0–1 law

$$R(A)\cup R(\emptyset)=R(A), \tag{46}$$

$$R(A)\cap R(U)=R(A). \tag{47}$$

Proof

$$R(A)\cup R(\emptyset)=(\underline{A}\cup\underline{\emptyset}\cup\underline{Z}_A(\emptyset),\overline{A}\cup\overline{\emptyset})=(\underline{A},\overline{A})=R(A),$$

$$R(A)\cap R(U)=(\underline{A}\cap\underline{U},\overline{A}\cap\overline{U}-\overline{Z}_A(U))=(\underline{A},\overline{A})=R(A). \qquad\square$$

Remark 4.1. $R(A)\cap R(\emptyset)=R(\emptyset)$ and $R(A)\cup R(U)=R(U)$ can also be proved easily.

(6) Complementary law

$$R(A)\cup R(\neg A)=R(U), \tag{48}$$

$$R(A)\cap R(\neg A)=R(\emptyset). \tag{49}$$

Proof

$$
\begin{aligned}
R(A) \cup R(\neg A) &= (\underline{A} \cup \underline{(\neg A)} \cup \underline{Z}_A(\neg A), \overline{A} \cup \overline{(\neg A)}) \\
&= (\underline{A} \cup \underline{(\neg A)} \cup \underline{Z}_A(\neg A), U) = (\underline{A} \cup (U - \overline{A}) \cup (\overline{A} - \underline{A}), U) \\
&= (U, U) = R(U),
\end{aligned}
$$

$$
\begin{aligned}
R(A) \cap R(\neg A) &= (\underline{A} \cap \underline{(\neg A)}, \overline{A} \cap \overline{(\neg A)} - \overline{Z}_A(\neg A)) \\
&= (\emptyset, \overline{A} \cap (U - \underline{A}) - (\overline{A} - \underline{A})) = (\emptyset, \emptyset) = R(\emptyset). \qquad \square
\end{aligned}
$$

Remark 4.2. It can also be proved that $\neg R(\neg A) = R(A)$. That is to say, $R(\neg A)$ is the complement to $R(A)$, and $R(A)$ is the complement to $R(\neg A)$.

(7) De Morgan law

$$\neg(R(A) \cup R(B)) = R(\neg A) \cap R(\neg B), \tag{50}$$

$$\neg(R(A) \cap R(B)) = R(\neg A) \cup R(\neg B). \tag{51}$$

Proof. If we want to prove that $R(\neg A) \cap R(\neg B)$ is the complement of $R(A) \cup R(B)$, we only need to prove

$$(R(A) \cup R(B)) \cap [R(\neg A) \cap R(\neg B)] = R(\emptyset),$$

$$(R(A) \cup R(B)) \cup [R(\neg A) \cap R(\neg B)] = R(U).$$

First, $(R(A) \cup R(B)) \cap [R(\neg A) \cap R(\neg B)] = R(\emptyset)$ since

$$
\begin{aligned}
(R(A) \cup R(B)) \cap [R(\neg A) \cap R(\neg B)] &= [(R(A) \cup R(B)) \cap R(\neg A)] \cap R(\neg B) \\
&= \{[R(A) \cap R(\neg A)] \cup [R(B) \cap R(\neg A)]\} \\
&\quad \cap R(\neg B) \\
&= \{[R(\emptyset) \cup [R(B) \cap R(\neg A)]\} \cap R(\neg B) \\
&= R(B) \cap R(\neg A) \cap R(\neg B) \\
&= R(B) \cap R(\neg B) \cap R(\neg A) \\
&= R(\emptyset) \cap R(\neg A) = R(\emptyset).
\end{aligned}
$$

Second, $(R(A) \cup R(B)) \cup [R(\neg A) \cap R(\neg B)] = R(U)$ since

$$\begin{aligned}(R(A)\cup R(B))\cup[R(\neg A)\cap R(\neg B)] &= [R(A)\cup R(B)\cup R(\neg A)]\cap[R(A)\cup R(B)\\ &\quad \cup R(\neg B)]\\ &= [R(A)\cup R(\neg A)\cup R(B)]\cap[R(A)\cup R(B)\\ &\quad \cup R(\neg B)]\\ &= (R(U)\cup R(B))\cap(R(A)\cup R(U))\\ &= R(U)\cap R(U)=R(U).\end{aligned}$$

Thus, $\neg(R(A)\cup R(B)) = R(\neg A)\cap R(\neg B)$ is proved. In the same way, we can also prove $\neg(R(A)\cap R(B)) = R(\neg A)\cup R(\neg B)$. Therefore, the De Morgan law is proved. $\square$

In the above, the seven basic properties of operations in rough sets have been introduced. In particular, the operation satisfies the Complementary law as shown. We can then discuss the algebraic property of rough sets.

5. Boolean algebra of rough sets

In this section, we will show that the collection of rough sets of an approximation space forms a Boolean algebra under the new operators introduced in the preceding sections. Such a result will be very helpful in studying the logic for rough sets [4,6,16–18,22]. Let U be the universe. Let X be an non-empty subset of U. In practice, we typically care for how to use the equivalence classes of R indicating the set X, i.e., rough set $R(X)$. Thus, it is reasonable to study X and $R(X)$ as a whole.

Definition 5.1. Let U be the universe. We call $2^U = \{X \mid X \subseteq U\}$ the power set on U. Define $J = \{(X, R(X)) \mid X \in 2^U\}$ to be a rough approximation space.

Theorem 5.1. *Let U be the universe. Let 2^U be the power set on U and J be a rough approximation space. There exists a one-to-one mapping from 2^U to J.*

Proof. Suppose $X \in 2^U$, define $g(X) = (X, R(X))$. Therefore, $g(X) \in J$. We have $g(X) = g(Y)$, if and only if $(X, R(X)) = (Y, R(Y))$, which yields $X = Y$. For any $(X, R(X)) \in J$, there exists $X \in 2^U$, which implies that $g(X) = (X, R(X))$. Thus, g is a one-to-one mapping. $\square$

Definition 5.2. Let U be the universe. Let 2^U be the power set on U. For any $X, Y \in 2^U$, the union, the intersection, and the complement operations in rough approximation space are given by

$$\begin{aligned}(X, R(X)) \cup (Y, R(Y)) &= (X \cup Y, R(X) \cup R(Y)) \\ &= (X \cup Y, R(X \cup Y))\end{aligned} \tag{52}$$

$$\begin{aligned}(X, R(X)) \cap (Y, R(Y)) &= (X \cap Y, R(X) \cap R(Y)) \\ &= (X \cap Y, R(X \cap Y))\end{aligned} \tag{53}$$

$$\neg(X, R(X)) = (\neg X, \neg R(X)) = (\neg X, R(\neg X)). \tag{54}$$

It is obvious that $X \cup Y$, $X \cap Y$, $\neg X$ belong to 2^U. Hence, $(X \cup Y, R(X \cup Y))$, $(X \cap Y, R(X \cap Y))$ and $(\neg X, R(\neg X))$ belong to J. This indicates that rough approximation space is closed for union, intersection, and complement operations. After defining the operations $\cup$, $\cap$, and $\neg$ in the rough approximation space, we can get an algebraic system $\langle J, \cup, \cap, \neg, 0, 1\rangle$. Here, 0 represents $(\emptyset, R(\emptyset))$, 1 represents $(U, R(U))$.

Theorem 5.2. *The algebraic system* $\langle J, \cup, \cap, \neg, 0, 1\rangle$ *is a Boolean algebra.*

Proof. It is easy to prove that the operations $\cup$ and $\cap$ in rough approximation space J satisfy the Commutative law, Associative law and Distributive law (the first Distributive law and the second Distributive law).

We will prove the 0–1 law.

For any $X \in 2^U$

$$(X, R(X)) \cup (\emptyset, R(\emptyset)) = (X \cup \emptyset, R(X) \cup R(\emptyset)) = (X, R(X)).$$

Thus, $(\emptyset, R(\emptyset))$ is the identity element of operation $\cup$, i.e., the 0 identity element of the Boolean algebra. For any $X \in 2^U$

$$\begin{aligned}(X, R(X)) \cap (U, R(U)) &= (X \cap U, R(X) \cap R(U)) = (X, R(X \cap U)) \\ &= (X, R(X)).\end{aligned}$$

Thus, $(U, R(U))$ is the identity element of operation $\cap$, i.e., the 1 identity element of the Boolean algebra.

Finally, we will prove the existence of the Boolean complement. For any $X \in 2^U$, $(X, R(X)) \in J$, $\neg(X, R(X)) = (\neg X, R(\neg X)) \in J$, we have

$$(X, R(X)) \cup (\neg X, R(\neg X)) = (X \cup (\neg X), R(X) \cup R(\neg X)) = (U, R(U)),$$

$$(X, R(X)) \cap (\neg X, R(\neg X)) = (X \cap (\neg X), R(X) \cap R(\neg X)) = (\emptyset, R(\emptyset)).$$

Hence, the Boolean complement exists.

Therefore, the algebraic system $\langle J, \cup, \cap, \neg, 0, 1\rangle$ is a Boolean algebra. $\square$

6. Roughness of fuzzy sets

A measure of roughness of crisp sets was introduced by Pawlak [8]. A measure of roughness of fuzzy sets defined in the partitioned domain and making use of the concept of a rough fuzzy set was introduced by Banerjee [1]. In this section, we will study the roughness of fuzzy sets based on the α-cut set and the β-cut set of a fuzzy set and establish some new properties using the two operators defined earlier.

Let U be the universe and R be an equivalence relation on U. We use $[x]_R$ to denote an equivalence class in R containing an element $x \in U$. Let $\mathbf{A} : U \to [0,1]$ be a fuzzy set in U. We use $\mathbf{A}(x), x \in U$, to denote the membership function that gives the degree of membership of x in $\mathbf{A}$. Let us consider parameters α and β, where $0 < \beta \leqslant \alpha \leqslant 1$. The α-cut set $\mathbf{A}_\alpha$ and the β-cut set $\mathbf{A}_\beta$ of the fuzzy set $\mathbf{A}$ are, respectively, defined as

$$\mathbf{A}_\alpha = \{x \in U \mid \mathbf{A}(\mathbf{x}) \geqslant \alpha\}, \tag{55}$$

$$\mathbf{A}_\beta = \{x \in U \mid \mathbf{A}(\mathbf{x}) \geqslant \beta\}. \tag{56}$$

Remark 6.1. Let $\mathbf{B}$ be another fuzzy set in U. We have

$$(\mathbf{A} \cup \mathbf{B})_\alpha = \mathbf{A}_\alpha \cup \mathbf{B}_\alpha, \tag{57}$$

$$(\mathbf{A} \cap \mathbf{B})_\alpha = \mathbf{A}_\alpha \cap \mathbf{B}_\alpha. \tag{58}$$

From the fuzzy set theory, we know that the α-cut set $\mathbf{A}_\alpha$ and the β-cut set $\mathbf{A}_\beta$ of the fuzzy set $\mathbf{A}$ are crisp sets. We have the following definition for lower and upper approximations of a fuzzy set.

Definition 6.1. The α-lower approximation and the β-upper approximation of the fuzzy set $\mathbf{A}$ are defined as follows:

$$\underline{\mathbf{A}_\alpha} = \{x \in U \mid [x]_R \subseteq \mathbf{A}_\alpha\} = \bigcup \{[x]_R \mid x \in U, [x]_R \subseteq \mathbf{A}_\alpha\}, \tag{59}$$

$$\overline{\mathbf{A}_\beta} = \{x \in U \mid [x]_R \cap \mathbf{A}_\beta \neq \emptyset\} = \bigcup \{[x]_R \mid x \in U, [x]_R \cap \mathbf{A}_\beta \neq \emptyset\}. \tag{60}$$

It may then be said that $\underline{\mathbf{A}_\alpha}$ ($\overline{\mathbf{A}_\beta}$) is the collection of objects in U with α (β) as the minimum degree of definite (possible) membership in the fuzzy set $\mathbf{A}$. In other words, α and β act as thresholds of definiteness and possibility, respectively, in the membership of the objects of U to the fuzzy set $\mathbf{A}$.

From Definition 6.1, the following theorem is obvious.

Theorem 6.1. *Let U be the universe and R be an equivalence relation. Let $\mathbf{A}, \mathbf{B} : U \to [0,1]$ be two fuzzy sets in U and $0 < \beta \leqslant \alpha \leqslant 1$. We have*

$$\underline{(\mathbf{A} \cup \mathbf{B})_\alpha} = \underline{\mathbf{A}_\alpha} \cup \underline{\mathbf{B}_\alpha} \cup \underline{Z}_{\mathbf{A}_\alpha}(\mathbf{B}_\alpha), \tag{61}$$

$$\overline{(\mathbf{A} \cup \mathbf{B})_\beta} = \overline{\mathbf{A}_\beta} \cup \overline{\mathbf{B}_\beta}, \tag{62}$$

$$\underline{(\mathbf{A} \cap \mathbf{B})_\alpha} = \underline{\mathbf{A}_\alpha} \cap \underline{\mathbf{B}_\alpha}, \tag{63}$$

$$\overline{(\mathbf{A}\cap\mathbf{B})_\beta} = \overline{\mathbf{A}_\beta}\cap\overline{\mathbf{B}_\beta} - \overline{Z}_{\mathbf{A}_\beta}(\mathbf{B}_\beta), \tag{64}$$

where $\underline{Z}_{\mathbf{A}_\alpha}(\mathbf{B}_\alpha), \overline{Z}_{\mathbf{A}_\beta}(\mathbf{B}_\beta)$ *are the certain increment operator of* $\mathbf{A}_\alpha$ *and the uncertain decrement operator of* $\mathbf{B}_\beta$, *respectively*.

Definition 6.2. A roughness measure $\rho_{\mathbf{A}}^{\alpha,\beta}$ of fuzzy set $\mathbf{A}$ in U with respect to α,β, where $0<\beta\leqslant\alpha\leqslant 1$, is defined as

$$\rho_{\mathbf{A}}^{\alpha,\beta} = 1-\frac{|\underline{\mathbf{A}_\alpha}|}{|\overline{\mathbf{A}_\beta}|}, \tag{65}$$

where $|X|$ denotes the cardinality of a set X.

Some properties, which have been discussed in [1], about the roughness of a fuzzy set are proper for this paper. Next, we will give some more properties by using the newly defined operators.

Property 6.1

$$\rho_{\mathbf{A}\cup\mathbf{B}}^{\alpha,\beta} = 1-\frac{|\underline{\mathbf{A}\cup\mathbf{B}_\alpha}|}{|\overline{\mathbf{A}\cup\mathbf{B}_\beta}|} = 1-\frac{|\underline{\mathbf{A}_\alpha}\cup\underline{\mathbf{B}_\alpha}\cup\underline{Z}_{\mathbf{A}_\alpha}(\mathbf{B}_\alpha)|}{|\overline{\mathbf{A}_\beta}\cup\overline{\mathbf{B}_\beta}|}, \tag{66}$$

Property 6.2

$$\rho_{\mathbf{A}\cap\mathbf{B}}^{\alpha,\beta} = 1-\frac{|\underline{\mathbf{A}\cap\mathbf{B}_\alpha}|}{|\overline{\mathbf{A}\cap\mathbf{B}_\beta}|} = 1-\frac{|\underline{\mathbf{A}_\alpha}\cap\underline{\mathbf{B}_\alpha}|}{|\overline{\mathbf{A}_\beta}\cap\overline{\mathbf{B}_\beta}-\overline{Z}_{\mathbf{A}_\beta}(\mathbf{B}_\beta)|}. \tag{67}$$

Proof of Properties 6.1 and 6.2. Replacing $\mathbf{A}$ using $\mathbf{A}\cup\mathbf{B}$ in Definition 6.2, we have

$$\rho_{\mathbf{A}\cup\mathbf{B}}^{\alpha,\beta} = 1-\frac{|\underline{\mathbf{A}\cup\mathbf{B}_\alpha}|}{|\overline{\mathbf{A}\cup\mathbf{B}_\beta}|}.$$

From (61) and (62), we have Property 6.1.

Replacing $\mathbf{A}$ using $\mathbf{A}\cap\mathbf{B}$ in Definition 6.2, we have

$$\rho_{\mathbf{A}\cap\mathbf{B}}^{\alpha,\beta} = 1-\frac{|\underline{\mathbf{A}\cap\mathbf{B}_\alpha}|}{|\overline{\mathbf{A}\cap\mathbf{B}_\beta}|}.$$

From (63) and (64), we get Property 6.2. □

Next we discuss the relation between the roughness measures of fuzzy set $\mathbf{A},\mathbf{B},\mathbf{A}\cap\mathbf{B},\mathbf{A}\cup\mathbf{B}$.

Property 6.3

$$\begin{aligned}\rho_{\mathbf{A}\cup\mathbf{B}}^{\alpha,\beta}|\mathbf{A}_\beta\cup\mathbf{B}_\beta| &= \rho_{\mathbf{A}}^{\alpha,\beta}|\overline{\mathbf{A}_\beta}| + \rho_{\mathbf{B}}^{\alpha,\beta}|\overline{\mathbf{B}_\beta}| - \rho_{\mathbf{A}\cap\mathbf{B}}^{\alpha,\beta}|\overline{\mathbf{A}_\beta}\cap\overline{\mathbf{B}_\beta}-\overline{Z}_{\mathbf{A}_\beta}(\mathbf{B}_\beta)| \\ &\quad - |\overline{Z}_{\mathbf{A}_\beta}(\mathbf{B}_\beta)| - |\underline{Z}_{\mathbf{A}_\alpha}(\mathbf{B}_\alpha)|.\end{aligned} \tag{68}$$

Proof. As for any finite set X, Y,

$$|X \cup Y| = |X| + |Y| - |X \cap Y|. \tag{69}$$

If $X \subseteq Y$, we have

$$|Y - X| = |Y| - |X|, \tag{70}$$

$$\begin{aligned}
\rho^{\alpha,\beta}_{\mathbf{A}\cup\mathbf{B}}|\overline{\mathbf{A}_\beta} \cup \overline{\mathbf{B}_\beta}| &= |\overline{\mathbf{A}_\beta} \cup \overline{\mathbf{B}_\beta}| - |\underline{\mathbf{A}_\alpha} \cup \underline{\mathbf{B}_\alpha} \cup \underline{Z}_{\mathbf{A}_\alpha}(\mathbf{B}_\alpha)| \\
&= |\overline{\mathbf{A}_\beta}| + |\overline{\mathbf{B}_\beta}| - |\overline{\mathbf{A}_\beta} \cap \overline{\mathbf{B}_\beta}| - |\underline{\mathbf{A}_\alpha} \cup \underline{\mathbf{B}_\alpha}| - |\underline{Z}_{\mathbf{A}_\alpha}(\mathbf{B}_\alpha)| \\
&\quad + |(\underline{\mathbf{A}_\alpha} \cup \underline{\mathbf{B}_\alpha}) \cap \underline{Z}_{\mathbf{A}_\alpha}(\mathbf{B}_\alpha)| \\
&= |\overline{\mathbf{A}_\beta}| + |\overline{\mathbf{B}_\beta}| - |\overline{\mathbf{A}_\beta} \cap \overline{\mathbf{B}_\beta}| - |\underline{\mathbf{A}_\alpha}| - |\underline{\mathbf{B}_\alpha}| + |\underline{\mathbf{A}_\alpha} \cap \underline{\mathbf{B}_\alpha}| \\
&\quad - |\underline{Z}_{\mathbf{A}_\alpha}(\mathbf{B}_\alpha)| + |(\underline{\mathbf{A}_\alpha} \cup \underline{\mathbf{B}_\alpha}) \cap \underline{Z}_{\mathbf{A}_\alpha}(\mathbf{B}_\alpha)|.
\end{aligned}$$

From Property 3.3, Property 3.6 and the definitions of $\rho^{\alpha,\beta}_{\mathbf{A}}$ and $\rho^{\alpha,\beta}_{\mathbf{B}}$, we have

$$\begin{aligned}
\rho^{\alpha,\beta}_{\mathbf{A}\cup\mathbf{B}}|\overline{\mathbf{A}_\beta} \cup \overline{\mathbf{B}_\beta}| &= |\overline{\mathbf{A}_\beta}| + |\overline{\mathbf{B}_\beta}| - (|\overline{\mathbf{A}_\beta} \cap \overline{\mathbf{B}_\beta}| - |\overline{Z}_{\mathbf{A}_\beta}(\mathbf{B}_\beta)|) - |\overline{Z}_{\mathbf{A}_\beta}(\mathbf{B}_\beta)| \\
&\quad - |\underline{\mathbf{A}_\alpha}| - |\underline{\mathbf{B}_\alpha}| + |\underline{\mathbf{A}_\alpha} \cap \underline{\mathbf{B}_\alpha}| - |\underline{Z}_{\mathbf{A}_\alpha}(\mathbf{B}_\alpha)| + |\emptyset| \\
&= |\overline{\mathbf{A}_\beta}| + |\overline{\mathbf{B}_\beta}| - |\overline{\mathbf{A}_\beta} \cap \overline{\mathbf{B}_\beta} - \overline{Z}_{\mathbf{A}_\beta}(\mathbf{B}_\beta)| - |\overline{Z}_{\mathbf{A}_\beta}(\mathbf{B}_\beta)| \\
&\quad - |\underline{\mathbf{A}_\alpha}| - |\underline{\mathbf{B}_\alpha}| + |\underline{\mathbf{A}_\alpha} \cap \underline{\mathbf{B}_\alpha}| - |\underline{Z}_{\mathbf{A}_\alpha}(\mathbf{B}_\alpha)| \\
&= \rho^{\alpha,\beta}_{\mathbf{A}}|\overline{\mathbf{A}_\beta}| + \rho^{\alpha,\beta}_{\mathbf{B}}|\overline{\mathbf{B}_\beta}| - \rho^{\alpha,\beta}_{\mathbf{A}\cap\mathbf{B}}|\overline{\mathbf{A}_\beta} \cap \overline{\mathbf{B}_\beta} - \overline{Z}_{\mathbf{A}_\beta}(\mathbf{B}_\beta)| \\
&\quad - |\overline{Z}_{\mathbf{A}_\beta}(\mathbf{B}_\beta)| - |\underline{Z}_{\mathbf{A}_\alpha}(\mathbf{B}_\alpha)|. \qquad \square
\end{aligned}$$

7. Conclusions

Many fields are widely effected by the two inequalities, which cannot result in exact results or good properties, of rough set theory. The certain increment operator and the uncertain decrement operator introduced in this paper are helpful to solve these problems. In this paper, we have got some satisfied results in several fields by using the two newly defined operators, but we still believe that they are only a starting point for further study. For example, they can be used to model and identify a hyperbolic fuzzy model [23]. Investigations in this direction may produce interesting and useful results.

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