Lie Algebraic Stability Analysis for Switched Systems With Continuous-Time and Discrete-Time Subsystems

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***Abstract*—We analyze stability for switched systems which are composed of both continuous-time and discrete-time subsystems. By considering a Lie algebra generated by all subsystem matrices, we show that if all continuous-time subsystems are Hurwitz stable, all discrete-time subsystems are Schur stable, and furthermore the obtained Lie algebra is solvable, then there is a common quadratic Lyapunov function for all subsystems and thus the switched system is exponentially stable under arbitrary switching. A numerical example is provided to demonstrate the result.**

***Index Terms*—Arbitrary switching, common quadratic Lyapunov functions, continuous-time, discrete-time, exponential stability, Lie algebra, switched systems.**

I. Introduction

In the last two decades, there has been increasing interest in stability analysis and controller design for switched systems; see the survey papers [1], [2], the recent book [3] and the references cited therein. It is pointed out in [1], [3] that there are three basic problems in stability analysis and design of switched systems: 1) find conditions for stability under arbitrary switching; 2) identify the limited but useful class of stabilizing switching laws; and 3) construct a stabilizing switching law.

In this paper, we focus our attention on the first problem. The motivation is very obvious: if a switched system is stable under arbitrary switching, then we gain more flexibility for considering other control specifications. There are several existing works on Problem 1), most of which deal with the case where the switched systems are composed of continuous-time subsystems. [4] showed that when all subsystems are stable and commutative pairwise, the switched linear system is stable under arbitrary switching. [5] extended this result from the commutation condition to a Lie algebraic condition. [6] showed that a class of switched symmetric systems are asymptotically stable under arbitrary switching since a common quadratic Lyapunov function, in the form of $V(x) = x^T x$, exists for all subsystems. [6]–[8] extended the consideration to stability analysis problems for switched systems composed of discrete-time subsystems.

Motivated by the observation that all these papers deal with switched systems composed of only continuous-time subsystems or only discrete-time ones, the authors considered in [9] the *new type* of switched systems which are composed of both continuous-time and discrete-time dynamical subsystems. As also pointed out in [9], it is very easy to find many applications involving such kind of switched systems. A typical example is a continuous-time plant controlled either by a physically implemented regulator or by a digitally implemented one together with a switching rule between them.

Example 1: Consider the continuous-time linear time-invariant (LTI) system described by $\dot{x}(t) = Ax(t) + Bu(t)$, where $x(t)$ is the continuous-time state, $u(t)$ is the control input in time domain, and A, B are constant matrices. Suppose that a stabilizing state feedback $u(t) = Kx(t)$ has been designed so that $A + BK$ is Hurwitz stable (all the eigenvalues of $A + BK$ are in the open left complex plane). It is known that in any computer-aided system, the controller is implemented in a discrete-time manner. When the sampling period is small enough, the closed-loop system can be viewed as a continuous-time system described by $\dot{x}(t) = (A + BK)x(t)$. When the sampling period does not have to be very small, we only need dealing with the value change on sampling points, and thus it is natural to consider the discrete-time system $x(k+1) = e^{(A+BK)\tau}x(k)$, where τ is the sampling period and $x(k) \triangleq x(k\tau)$. Although we used the same feedback gain K here for simplicity, we may want to design different gains for continuous-time domain and discrete-time one. Therefore, the entire system can be considered as a switched system composed of continuous-time and discrete-time subsystems. ∎

For several cases of such mixed-type switched systems, [9] has given some analysis and design results. For example, in the case where commutation condition holds, and the case of switched symmetric systems, it is shown that if all subsystems are (Hurwitz or Schur) stable, then there exists a common quadratic Lyapunov function for all subsystems and thus the switched system is exponentially stable under arbitrary switching. Recently, the authors extended the results for switched symmetric systems in [9] to switched *normal* systems in [10]. For such switched systems, it is shown that when all continuous-time subsystems are Hurwitz stable and all discrete-time subsystems are Schur stable, a common quadratic Lyapunov function exists for the subsystems and that

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the switched system is exponentially stable under arbitrary switching. Some discussions are also given for the case where unstable subsystems are involved.

In this paper, we aim to apply the Lie algebraic approach, which was proposed in [5], [11], for switched systems composed of both continuous-time and discrete-time subsystems. By considering a Lie algebra generated by all subsystem matrices, we show that if all continuous-time subsystems are Hurwitz stable and all discrete-time subsystems are Schur stable, and furthermore the Lie algebra is solvable, then there is a common quadratic Lyapunov function for all subsystems and thus the switched system is exponentially stable under arbitrary switching. We establish the common quadratic Lyapunov function in a constructive way, and provide a numerical example to demonstrate the result.

II. System Description & Preliminaries

We consider the switched system which is composed of a set of continuous-time LTI subsystems

$$\dot{x}(t) = A_{ci}x(t), \qquad i = 1, \ldots, N_c \tag{1}$$

and a set of discrete-time LTI subsystems

$$x(k+1) = A_{dj}x(k), \qquad j = 1, \ldots, N_d \tag{2}$$

where $x(t)$, $x(k) \in \Re^n$ are the subsystem states, A_{ci}'s and A_{dj}'s are constant matrices of appropriate dimension. To discuss stability of the overall switched system, we assume for simplicity but without loss of generality that the sampling periods of all the discrete-time subsystems are of the same value τ (the discussion can be easily extended to the case where the discrete-time subsystems have different sampling periods). Since the states of the discrete-time subsystems can be viewed as piecewise constant vectors between sampling points, we can consider the value of the system states in the continuous-time domain. For example, if subsystem A_{c1} is activated on $[t_0, t_1]$ and then subsystem A_{d1} is activated for m steps and subsystem A_{c2} is activated from then to t_2, the continuous-time domain is divided into

$$[t_0, t_2] = [t_0, t_1] \cup [t_1, t_1 + m\tau] \cup [t_1 + m\tau, t_2] \tag{3}$$

and the system state is defined to take the form shown in (4) at the bottom of the page. Although $x(t)$ is not continuous with respect to time t due to existence of discrete-time subsystems, the solution $x(t)$ is uniquely defined at all time instants, and thus various stability properties can be discussed in the continuous-time domain.

At the end of this section, we give some preliminaries of Lie algebra for integrity. Most of the material is picked up from [5], [11]. Interested readers are referred to these references or more detailed textbooks on Lie algebras [12], [13].

A Lie algebra $\mathcal{L}$ is a finite-dimensional vector space equipped with a Lie bracket, i.e., a bilinear, skew-symmetric map $\mathcal{L} \times \mathcal{L} \to \mathcal{L}$ satisfying the Jacobi identity $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] = 0$. In the case of matrix Lie algebra, the standard Lie bracket is defined as $[A,B] \triangleq AB - BA$. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are linear subspaces of a Lie algebra $\mathcal{L}$, we write $[\mathcal{L}_1, \mathcal{L}_2]$ for the linear space spanned by all the products $[L_1, L_2]$ with $L_1 \in \mathcal{L}_1$ and $L_2 \in \mathcal{L}_2$, and we define the sequence $\mathcal{L}^{(k)}$ inductively as follows:

$$\mathcal{L}^{(1)} \triangleq \mathcal{L}, \quad \mathcal{L}^{(k+1)} \triangleq [\mathcal{L}^{(k)}, \mathcal{L}^{(k)}] \subset \mathcal{L}^{(k)}. \tag{5}$$

If $\mathcal{L}^{(k)} = 0$ for some k sufficiently large, then $\mathcal{L}$ is called *solvable*. For example, if $\mathcal{L}$ is a Lie algebra generated by two matrices A and B, we have

$$\begin{aligned} \mathcal{L}^{(1)} &= \mathrm{span}\{A, B, [A,B], [A,[A,B]], \ldots\} \\ \mathcal{L}^{(2)} &= \mathrm{span}\{[A,B], [A,[A,B]], [B,[A,B]], \ldots\} \\ \mathcal{L}^{(3)} &= \mathrm{span}\{[[A,B],[A,[A,B]]], \ldots\} \end{aligned} \tag{6}$$

and so on.

The following result plays a key role in our subsequent discussion. It is known as Lie's Theorem and can be found in most textbooks on Lie algebra theory [12], [13].

Lemma 1: Let $\mathcal{L}$ be a solvable Lie algebra over an algebraically closed field, and let ρ be a representation of $\mathcal{L}$ on a vector space V of finite dimension n. Then, there exists a basis $\{v_1, v_2, \ldots, v_n\}$ of V such that for each $X \in \mathcal{L}$ the matrix $\rho(X)$ in that basis takes the upper-triangular form

$$\begin{bmatrix} \lambda_1(X) & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n(X) \end{bmatrix} \tag{7}$$

where $\lambda_1(X), \ldots, \lambda_n(X)$ are its eigenvalues. ∎

This lemma will be used in Section IV for the Lie algebra composed of all subsystem matrices in the switched system.

III. Stability Analysis Using Common Quadratic Lyapunov Function

In this section, we discuss the switched system's stability using the approach of common quadratic Lyapunov functions.

Definition 1: If there is a common positive definite matrix P satisfying

$$A_{ci}^T P + P A_{ci} < 0, \qquad i = 1, \ldots, N_c \tag{8}$$

$$A_{dj}^T P A_{dj} - P < 0, \qquad j = 1, \ldots, N_d \tag{9}$$

then $V(x) = x^T P x$ is called a *common quadratic Lyapunov function* (CQLF) for all the subsystems.

Remark 1: There are many switched systems in which all the subsystems have a CQLF. For example, we have shown [6] that if all the subsystems are (Hurwitz or Schur) stable and symmetric, then $V(x) = x^T x$ $(P = I)$ is a CQLF. The result has

$$x(t) = \begin{cases} e^{A_{c1}(t-t_0)}x(t_0), & t \in [t_0, t_1] \\ A_{d1}^k x(t_1), & t \in (t_1 + k\tau, t_1 + (k+1)\tau], 1 \le k \le m-1 \\ e^{A_{c2}(t-t_1-m\tau)}x(t_1 + m\tau), & t \in (t_1 + m\tau, t_2] \end{cases} \tag{4}$$

been extended in [10] to switched normal systems (that satisfy $A_{ci}^T A_{ci} = A_{ci} A_{ci}^T$, $A_{dj}^T A_{dj} = A_{dj} A_{dj}^T$). In [9], we have proved constructively that if all the subsystems are (Hurwitz or Schur) stable and commutative pariwise, then there exists a CQLF for all the subsystems. ∎

Theorem 1: If there is a CQLF for all the subsystems, the switched system composed of (1) and (2) is exponentially stable under arbitrary switching.

Proof: To show the exponential stability of the system, we first find two positive scalars α_c and $\alpha_d < 1$ such that

$$\begin{aligned} A_{ci}^T P + P A_{ci} &< -2\alpha_c P \\ A_{dj}^T P A_{dj} - \alpha_d^2 P &< 0 \end{aligned} \tag{10}$$

hold for all i and j. Then, in the period where a continuous-time subsystem is activated, we obtain $\dot{V}(x(t)) < -2\alpha_c V(x(t))$, and in the period where a discrete-time subsystem is activated, $V(x(k+1)) < \alpha_d^2 V(x(k))$.

For any time $t > 0$ (when a discrete-time subsystem is active at t, we refer to t tacitly as the last sampling point since the state does not change until the next sampling point), we can always divide the time interval [0, t] as $t = t_c + m\tau$ ($m \geq 0$), where t_c is the total duration time on continuous-time subsystems and $m\tau$ is the total duration time on discrete-time subsystems. It is not difficult to obtain that no matter what the activation order is

$$V(x(t)) \leq e^{-2\alpha_c t_c} \alpha_d^{2m} V(x(0)) \tag{11}$$

and thus

$$|x(t)| \leq \sqrt{\frac{\lambda_M(P)}{\lambda_m(P)}} e^{-\alpha t} |x(0)| \tag{12}$$

where $\alpha = \min\{\alpha_c, \ln(\alpha_d^{-1})/\tau\} > 0$, $\lambda_M(P)$ and $\lambda_m(P)$ denote the largest and the smallest eigenvalue of P, respectively. Since we did not add any limitation on the switching signals, the switched system is exponentially stable under arbitrary switching. ∎

IV. Lie Algebraic Condition

We first state a result which will be used later in the proof of the main theorem.

Lemma 2: All leading principal minors of a Hermitian matrix are real. A Hermitian matrix H is positive definite (i.e., $x^* H x > 0$, $\forall x \neq 0$) if and only if all its leading principal minors are positive. ∎

Now, we state and prove the main theorem of this paper. It is noted that the idea of the proof was proposed in [5], where the switched systems composed of only continuous-time subsystems are discussed.

Theorem 2: If all continuous-time subsystems are Hurwitz stable and all discrete-time subsystems are Schur stable, and furthermore the Lie algebra

$$\left\{ A_{ci},\ i = 1, \ldots, N_c;\ A_{dj},\ j = 1, \ldots, N_d \right\}_{LA} \tag{13}$$

is solvable, then the switched system composed of (1) and (2) is exponentially stable under arbitrary switching.

Proof: According to Theorem 1, the proof is reduced to finding a CQLF for all the subsystems. Without loss of generality, we assume here for simplicity that the dimension of the matrices A_{ci} and A_{dj} is $n = 3$.

Translating Lemma 1 into the present situation, we see that if the Lie algebra (13) is solvable, then there exists a nonsingular complex matrix U such that for all i, j

$$A_{ci} = U^{-1} \tilde{A}_{ci} U, \qquad A_{dj} = U^{-1} \tilde{A}_{dj} U \tag{14}$$

where the complex matrices $\tilde{A}_{ci}$, $\tilde{A}_{dj}$ are upper-triangular.

We first show that there exists a real positive definite matrix $\tilde{P}$ such that

$$\tilde{A}_{ci}^* \tilde{P} + \tilde{P} \tilde{A}_{ci} < 0, \quad \tilde{A}_{dj}^* \tilde{P} \tilde{A}_{dj} - \tilde{P} < 0. \tag{15}$$

Especially, we choose $\tilde{P}$ with real diagonal form as $\tilde{P} = \text{diag}\{\tilde{p}_1, \tilde{p}_2, \tilde{p}_3\}$, and thus we have

$$-\tilde{A}_{ci}^* \tilde{P} - \tilde{P} \tilde{A}_{ci} = \begin{bmatrix} -2\tilde{p}_1(\Re \tilde{A}_{ci})_{11} & -\tilde{p}_1(\tilde{A}_{ci})_{12} & -\tilde{p}_1(\tilde{A}_{ci})_{13} \\ * & -2\tilde{p}_2(\Re \tilde{A}_{ci})_{22} & -\tilde{p}_2(\tilde{A}_{ci})_{23} \\ * & * & -2\tilde{p}_3(\Re \tilde{A}_{ci})_{33} \end{bmatrix} \tag{16}$$

and (17), shown at the bottom of the page. Since all the subsystems are assumed to be Hurwitz/Schur stable, $(\Re \tilde{A}_{ci})_{11} < 0$ and $|(\tilde{A}_{dj})_{11}| < 1$ hold for all i's and j's. Then, we can choose the positive scalar $\tilde{p}_1$ arbitrarily so that the first leading principal minors of (16) and (17), namely, $-2\tilde{p}_1(\Re \tilde{A}_{ci})_{11}$ and $\tilde{p}_1(1 - |(\tilde{A}_{dj})_{11}|^2)$, are positive.

Next, since $(\Re \tilde{A}_{ci})_{22} < 0$ and $|(\tilde{A}_{dj})_{22}| < 1$, we can always find a positive scalar $\tilde{p}_2$ sufficiently large such that for the fixed $\tilde{p}_1$, the second leading principal minors of (16) and (17), namely

$$\begin{vmatrix} -2\tilde{p}_1(\Re \tilde{A}_{ci})_{11} & -\tilde{p}_1(\tilde{A}_{ci})_{12} \\ * & -2\tilde{p}_2(\Re \tilde{A}_{ci})_{22} \end{vmatrix} \tag{18}$$

and

$$\begin{vmatrix} \tilde{p}_1(1 - |(\tilde{A}_{dj})_{11}|^2) & -\tilde{p}_1(\tilde{A}_{dj})_{11}^*(\tilde{A}_{dj})_{12} \\ * & \tilde{p}_2(1 - |(\tilde{A}_{dj})_{22}|^2) - \tilde{p}_1|(\tilde{A}_{dj})_{12}|^2 \end{vmatrix} \tag{19}$$

are positive.

Finally, since $(\Re \tilde{A}_{ci})_{33} < 0$ and $|(\tilde{A}_{dj})_{33}| < 1$, for the fixed $\tilde{p}_1$ and $\tilde{p}_2$, we can always find a positive scalar $\tilde{p}_3$ sufficiently large such that the third leading principal minors (i.e., the determinants) of (16) and (17) are positive.

In this way, we have chosen $\tilde{p}_1$, $\tilde{p}_2$ and $\tilde{p}_3$ orderly so that all the leading principal minors of (16) and (17) are positive. There-

$$-\tilde{A}_{dj}^* \tilde{P} \tilde{A}_{dj} + \tilde{P} = \begin{bmatrix} \tilde{p}_1(1 - |(\tilde{A}_{dj})_{11}|^2) & -\tilde{p}_1(\tilde{A}_{dj})_{11}^*(\tilde{A}_{dj})_{12} & -\tilde{p}_1(\tilde{A}_{dj})_{11}^*(\tilde{A}_{dj})_{13} \\ * & \tilde{p}_2(1 - |(\tilde{A}_{dj})_{22}|^2) - \tilde{p}_1|(\tilde{A}_{dj})_{12}|^2 & -\tilde{p}_1(\tilde{A}_{dj})_{12}^*(\tilde{A}_{dj})_{13} - \tilde{p}_2(\tilde{A}_{dj})_{22}^*(\tilde{A}_{dj})_{23} \\ * & * & \tilde{p}_3(1 - |(\tilde{A}_{dj})_{33}|^2) - \tilde{p}_1|(\tilde{A}_{dj})_{13}|^2 - \tilde{p}_2|(\tilde{A}_{dj})_{23}|^2 \end{bmatrix} \tag{17}$$

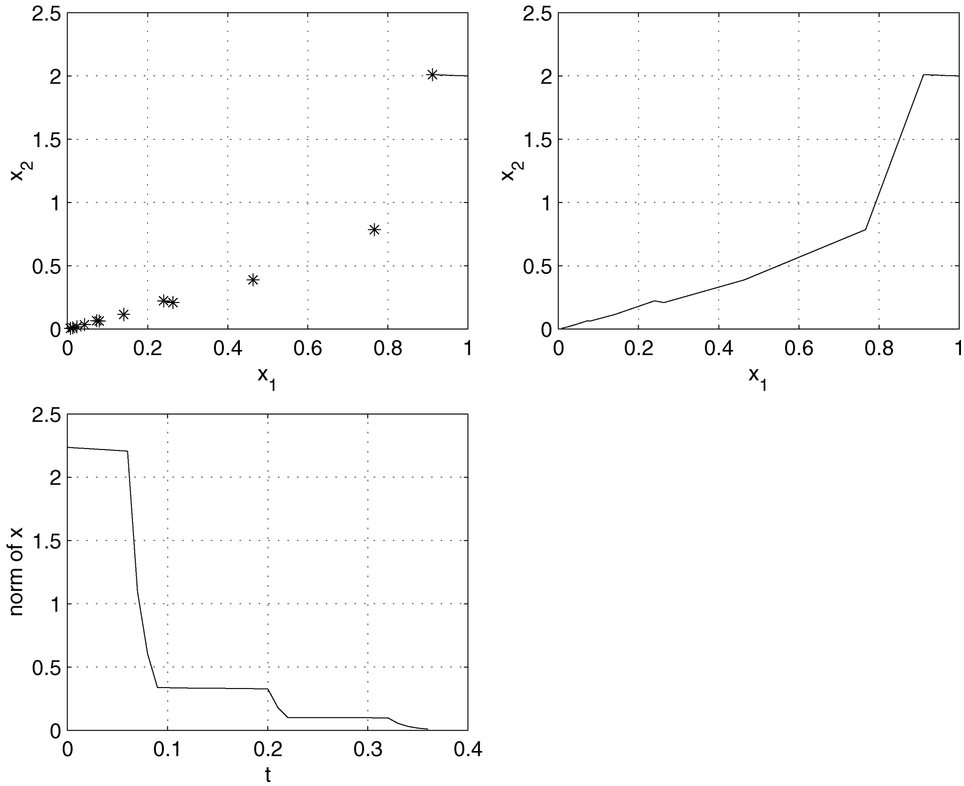

Fig. 1. System's state and its norm in Example 1.

fore, according to Lemma 2, (15) is satisfied with the chosen $\tilde{P}$. Using the obtained $\tilde{P}$, we substitute (14) into (15) to obtain

$$\begin{aligned}&\tilde{P}UA_{ci}U^{-1} + (U^{-1})^* A_{ci}^T U^* \tilde{P} < 0\\ &(U^{-1})^* A_{dj}^T U^* \tilde{P} U A_{dj} U^{-1} - \tilde{P} < 0\end{aligned} \quad (20)$$

which are respectively equivalent to

$$PA_{ci} + A_{ci}^T P < 0, \quad A_{dj}^T P A_{dj} - P < 0 \quad (21)$$

where $P = U^* \tilde{P} U$.

We write the complex matrix P as $P = \Re(P) + \sqrt{-1}\Im(P)$. Since P is Hermitian, $\Im(P)$ is skew-symmetric, from which $x^T P x = x^T \Re(P) x > 0$ $(x \neq 0)$ is obtained. Thus, $\Re(P)$ is a real positive definite matrix. Similarly, we can obtain easily

$$A_{ci}^T \Re(P) + \Re(P) A_{ci} < 0,\ A_{dj}^T \Re(P) A_{dj} - \Re(P) < 0 \quad (22)$$

which implies that $\Re(P)$ is the common Lyapunov matrix we want to compute. This completes the proof. ∎

It can be seen from the above proof that the important issue used for stability under arbitrary switching is all the subsystems are simultaneously triangularised by a nonsingular complex matrix (see also Remark 4). The Lie algebra condition is only sufficient for this condition. In this sense, the theorem can be stated and applied in more general case.

Remark 2: It is understood from the proof of Theorem 2 that the result can be extended to case where both upper-triangular and lower-triangular $\tilde{A}_{ci}$'s (or $\tilde{A}_{dj}$'s) exist. This means that if the Lie algebra (13) is not solvable, we can try to replace some subsystem matrices with their transposes in (13) and then check the new Lie algebra again. ∎

Remark 3: As also pointed out in [5], although we have showed the existence condition of CQLF constructively, the computation depends on the value of the transformation matrix U. Since it may need some efforts obtaining the value of U when using standard numerical methods, it may be more efficient to solve the LMI's (8)–(9) with respect to $P > 0$ directly, using the existing LMI softwares. ∎

Remark 4: Another method of proving the stability part of Theorem 2 originated from [14], [15], where the main approach is to find a real nonsingular matrix G so that all the subsystem matrices are transformed into

$$G^{-1} A_{\triangle} G = \begin{bmatrix} A_{1\triangle} & * & \cdots & * \\ 0 & A_{2\triangle} & \cdots & * \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & A_{l\triangle} \end{bmatrix} \quad (23)$$

where $A_{m\triangle}$ $(1 \leq m \leq l)$ is either a real scalar $\alpha_{m\triangle}$ or a 2×2 matrix taking the form of $\begin{bmatrix} \mu_{m\triangle} & \omega_{m\triangle} \\ -\omega_{m\triangle} & \mu_{m\triangle} \end{bmatrix}$. Then, for the continuous-time subsystems that are Hurwitz stable, we have $\alpha_{m\triangle} < 0$, $\mu_{m\triangle} < 0$, and for the discrete-time subsystems that are Schur stable, we have $|\alpha_{m\triangle}| < 1$, $\mu_{m\triangle}^2 + \omega_{m\triangle}^2 < 1$. The remaining proof can be done similarly as in [14], [15]. ∎

Now, we give an example to demonstrate the main result.

Example 2: Consider the switched system with one continuous-time subsystem and one discrete-time subsystem whose system matrices are

$$A_c = \begin{bmatrix} -0.5 & -0.5 \\ 0.1 & -0.3 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0.4 & 0.2 \\ 0.2 & 0.3 \end{bmatrix}. \tag{24}$$

It is easy to confirm that A_c is Hurwitz stable and A_d is Schur stable.

Some standard Lie brackets are computed as

$$\begin{aligned} [A_c, A_d] &= \begin{bmatrix} -0.1200 & 0.0100 \\ 0.0500 & 0.1200 \end{bmatrix} \\ [A_c, [A_c, A_d]] &= \begin{bmatrix} -0.0260 & -0.1220 \\ -0.0140 & 0.0260 \end{bmatrix} \\ [A_d, [A_c, A_d]] &= \begin{bmatrix} -0.0080 & 0.0490 \\ -0.0530 & -0.0080 \end{bmatrix} \\ [[A_c, A_d], [A_c, [A_c, A_d]]] &= \begin{bmatrix} -0.0060 & -0.0298 \\ -0.0060 & 0.0060 \end{bmatrix} \\ [[A_c, A_d], [A_d, [A_c, A_d]]] &= \begin{bmatrix} -0.0030 & -0.0119 \\ -0.0119 & 0.0030 \end{bmatrix} \end{aligned} \tag{25}$$

and

$$\begin{aligned} &[[[A_c, A_d], [A_c, [A_c, A_d]]], [[A_c, A_d], [A_d, [A_c, A_d]]]] \\ &\qquad = \begin{bmatrix} -0.0004 & 0.0000 \\ 0.0001 & 0.0004 \end{bmatrix}. \end{aligned} \tag{26}$$

It is seen from further computation that the Lie algebra $\{A_c, A_d\}_{LA}$ is solvable with $k = 5$, and thus the switched system is exponentially stable under arbitrary switching.

Suppose that the sampling period of subsystem A_d is 0.01. Fig. 1 shows the convergence of the system state and the norm when A_c and A_d are activated alternatively with a randomly generated time series (0.06, 3 steps, 0.11, 2 steps, 0.09, 4 steps). The initial state is $[1 \;\; 2]^T$, and the mark "*" in the upper-left part of Fig. 1 describes the discrete-time state change, while the upper-right part of Fig. 1 connects all the sampling points into a continuous trajectory. The lower part of Fig. 1 shows that the norm of the system state converges to zero quickly.

We note finally that the computation of common Lyapunov function (matrix) has much more importance when nonlinear terms are involved. For example, consider the perturbed switched system composed of

$$\dot{x}(t) = A_{ci}x(t) + f_{ci}(x,t), \qquad i = 1, \ldots, N_c \tag{27}$$

and

$$x(k+1) = A_{dj}x(k) + f_{dj}(x,k), \qquad j = 1, \ldots, N_d \tag{28}$$

where $f_{ci}(x,t)$'s and $f_{dj}(x,k)$'s denote the perturbations. Suppose that the split between $A_{ci}x(t)$ (or $A_{dj}x(k)$) and $f_{ci}(x,t)$ (or $f_{dj}(x,k)$), is done so that A_{ci}'s are Hurwitz stable, A_{dj}'s are Schur stable, and the Lie algebra (13) is solvable. Let P_f be the obtained common Lyapunov matrix. Then, the switched system remains exponentially stable provided

$$x^T(t)P_f f_{ci}(x(t),t) \le 0, \;\; x^T(k)P_f f_{dj}(x(k),k) \le 0. \tag{29}$$

Noticing that we still have much freedom in constructing the common Lyapunov matrix in the proof of Theorem 2, we may consider turning back to repeat the computation procedure if the above inequalities are not true.

V. Conclusion

A Lie algebraic condition has been established for stability analysis of a *new* class of switched systems, which are composed of both continuous-time and discrete-time linear subsystems, under *arbitrary* switching. The result is theoretically attractive, and the computation is not involved. Future work includes its extention to the case of general nonlinear subsystems, and to the case where unstable subsystems are involved.

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