Necessary and Sufficient Conditions for the Hurwitz and Schur Stability of Interval Matrices

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Abstract—**We establish a set of new sufficient conditions for the Hurwitz and Schur stability of interval matrices. We use these results to establish necessary and sufficient conditions for the Hurwitz and Schur stability of interval matrices. We relate the above results to the existence of quadratic Lyapunov functions for linear time-invariant systems with interval-valued coefficient matrices. Using the above results, we develop an algorithm to determine the Hurwitz and the Schur stability properties of interval matrices. We demonstrate the applicability of our results by means of two specific examples.**

I. Introduction

The stability of interval matrices is of great current interest (see, e.g., [1], [2], [4]–[6]). Most of the existing results constitute sufficient conditions for stability, and some of them provide necessary conditions for the stability of interval matrices. The very few results which offer necessary *and* sufficient stability conditions are concerned with low order cases or involve criteria which are not practical to check (see, e.g., [1]).

In the present paper, we establish necessary and sufficient conditions for the Hurwitz stability and for the Schur stability of interval matrices. We also present an algorithm which enables us to verify these stability results.

In the next section, we identify the notation used throughout this paper. In Section 3, we first establish sufficient conditions for the Hurwitz and Schur stability of interval matrices (Lemmas 1a and 1b). These results are used in Section IV to determine necessary and sufficient conditions for Hurwitz and Schur stability of interval matrices (Lemmas 2a and 2b and Theorems 1a and 1b). In Section V, we consider quadratic Lyapunov functions and we establish a connection between the concepts of Hurwitz (Schur) stability and quadratic Hurwitz (Schur) stability (Corollary 1 and Theorem 2).

Using the above results, we develop in Section VI an algorithm which verifies the Hurwitz and Schur stability properties of interval matrices. We demonstrate the applicability of our results by means of two specific examples (Examples 1 and 2).

II. Notation

Let R^n denote real n-space. If $x \in R^n$, then $x^T = (x_1, \cdots, x_n)$ denotes the transpose of x. Let $R^{n\times m}$ denote the set of $n \times m$ real matrices. If $B = [b_{ij}]_{n\times m} \in R^{n\times m}$, then B^T denotes the transpose of B.

For $A^m = [a_{ij}^m]_{n\times n}$ and $A^M = [a_{ij}^M]_{n\times n}$ satisfying $a_{ij}^m \le a_{ij}^M$ for all $1 \le i, j \le n$, we define the interval matrix $[A^m, A^M]$ by $[A^m, A^M] = \{A = [a_{ij}]: a_{ij}^m \le a_{ij} \le a_{ij}^M, 1 = i, j \le n\}$. If for another interval matrix $[A_0^m, A_0^M]$ it is true that $[A_0^m, A_0^M] \subset [A^m, A^M]$, we call $[A_0^m, A_0^M]$ a subinterval matrix of the interval matrix $[A^m, A^M]$. Frequently, in the interests of brevity, we will refer to an interval matrix simply as an interval and to a subinterval matrix as a subinterval. Also, for $A \in R^{n\times n}$ and $\Delta A \in R^{n\times n}$, where all elements of ΔA are nonnegative, we use the notation $[A \pm \Delta A]$ to represent the interval matrix $[A - \Delta A, A + \Delta A]$. For any interval matrix $[A^m, A^M]$, there is a unique representation of the form $[A \pm \Delta A]$. Indeed, A and ΔA are given by $A = (1/2)(A^m + A^M)$ and $\Delta A = (1/2)(A^M - A^m)$.

If for all $A \in [A^m, A^M]$, A is Hurwitz stable (i.e., all eigenvalues of A have negative real parts), we say that $[A^m, A^M]$ is Hurwitz stable. Similarly, if for all $A \in [A^m, A^M]$, A is Schur stable (i.e., all eigenvalues of A have magnitude less than one), we say that $[A^m, A^M]$ is Schur stable.

We let $|\cdot|$ denote any one of the equivalent norms on R^n. In particular, the norms $|\cdot|_p, 1 \le p \le \infty$ are defined by

$$|x|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}, \; 1 \le p < \infty \quad \text{and} \quad |x|_\infty = \max_{1\le i\le n} |x_i|$$

where $x = (x_1, \cdots, x_n)^T$.

The matrix norms $|\cdot|_p, 1 \le p \le \infty$, defined on $R^{n\times n}$ and induced by the norms $|\cdot|_p$ on $R^n, 1 \le p \le \infty$ are defined as

$$|A|_p = \sup_{0\ne x\in R^n} \frac{|Ax|_p}{|x|_p}, 1 \le p \le \infty.$$

In particular, we have

$$|A|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}| \quad \text{and} \quad |A|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$$

where $A = [a_{ij}]_{n\times n} \in R^{n\times n}$.

On the real linear space $R^{n\times n}$, we define the function $\alpha: R^{n\times n} \to [0, \infty)$ by

$$\alpha(A) = \max\{|A|_1, |A|_\infty\}. \tag{2.1}$$

It is not hard to show that $\alpha(\cdot)$ defines a norm on $R^{n\times n}$. We note, however, that $\alpha(\cdot)$ does not satisfy the property $\alpha(AB) \le \alpha(A) \cdot \alpha(B)$.

III. Sufficient Conditions

We first establish sufficient conditions for the Hurwitz stability and for the Schur stability of interval matrices. To this end, we will require the following hypotheses.

Assumption 1a: For $[A^m, A^M]$ we assume that

i) $A_0 \triangleq (1/2)(A^m + A^M)$ is Hurwitz stable, and therefore, there exists a positive definite matrix $P = P^T$ which is determined by the matrix equation

$$PA_0 + A_0^T P = -I \tag{3.1}$$

where $I \in R^{n\times n}$ denotes the identity matrix; and

ii) $$\alpha(A^M - A^m) < (1/|P|_\infty) \tag{3.2}$$

where $\alpha(\cdot)$ denotes the norm defined in (2.1). □

Assumption 1b: For $[A^m, A^M]$ we assume that

i) $A_0 \triangleq (1/2)(A^m + A^M)$ is Schur stable, and therefore, there exists a positive definite matrix $P = P^T$ which is determined by the matrix equation

$$A_0^T P A_0 - P = -I \tag{3.3}$$

where $I \in R^{n\times n}$ denotes the identity matrix; and

ii) $$\tfrac{1}{2}\alpha(A^M - A^m) < \left[\alpha(A_0)^2 + \tfrac{1}{|P|_\infty}\right]^{1/2} - \alpha(A_0) \tag{3.4}$$

where $\alpha(\cdot)$ denotes the norm defined in (2.1). □

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We first determine sufficient conditions for Hurwitz stability.

Lemma 1a: If $[A^m, A^M]$ satisfies Assumption 1a, then $[A^m, A^M]$ is Hurwitz stable.

Proof: It suffices to show that for all $A \in [A^m, A^M]$, the trivial solution $x = 0$ of

$$\dot{x} = Ax \tag{3.5}$$

is asymptotically stable, i.e., for $v(x) = x^T Px$, where P is determined by (3.1)

$$\dot{v}_{(3.5)}(x) = x^T(PA + A^T P)x < 0 \tag{3.6}$$

for all $x \neq 0$, and for all $A \in [A^m, A^M]$.

Let $\Delta A = A - A_0 = A - (1/2)(A^m + A^M)$. Then $\Delta A = [\Delta a_{ij}]_{n\times n}$ satisfies the relation

$$|\Delta a_{ij}| = |a_{ij} - \frac{1}{2}(a_{ij}^m + a_{ij}^M)| \leq \frac{1}{2}(a_{ij}^M - a_{ij}^m) \tag{3.7}$$

for all $1 \leq i, j \leq n$, where we have used the fact that $a_{ij}^m \leq a_{ij} \leq a_{ij}^M$, and where $|\cdot|$ denotes absolute value. We note that $(1/2)(a_{ij}^M - a_{ij}^m)$ is the (i, j)th element of $(1/2)(A^M - A^m)$ and (3.7) implies that

$$|\Delta A|_1 \leq \frac{1}{2}|A^M - A^m|_1 \text{and} |\Delta A|_\infty \leq \frac{1}{2}|A^M - A^m|_\infty.$$

Thus, we have

$$\alpha(\Delta A) \leq \frac{1}{2}\alpha(A^M - A^m) \tag{3.8}$$

where $\alpha(\cdot)$ is the norm defined in (2.1). From (3.8) and (3.2) we now obtain

$$\alpha(\Delta A) < \frac{1}{2|P|_\infty}. \tag{3.9}$$

We now show that (3.9) implies that $PA + A^T P$ is negative definite for all $A \in [A^m, A^M]$, and thus, (3.6) holds for all $A \in [A^m, A^M]$. We have

$$\begin{aligned} PA + A^T P &= P(A_0 + \Delta A) + (A_0 + \Delta A)^T P \\ &= PA_0 + A_0^T P + P(\Delta A) + (\Delta A)^T P \\ &= -I + P(\Delta A) + (\Delta A)^T P. \end{aligned} \tag{3.10}$$

To show that $PA + A^T P$ is negative definite, by (3.10) we only need to show that the largest eigenvalue of $P(\Delta A) + (\Delta A)^T P$ is less than one. To accomplish this, we will use the fact that for any square matrix B, $|\lambda(B)| \leq |B|_\infty$, where $\lambda(B)$ denotes any eigenvalue of B. Indeed, there is a vector $x_0 \neq 0$ such that $Bx_0 = \lambda(B)x_0$, and thus $|Bx_0|_\infty = |\lambda(B)|\,|x_0|_\infty$. Hence

$$|\lambda(B)| = \frac{|Bx_0|_\infty}{|x_0|_\infty} \leq |cB|_\infty. \tag{3.11}$$

Therefore, by (3.11), to show that the largest eigenvalue of $P(\Delta A) + (\Delta A)^T P$ is less than one, it suffices to show that

$$|P(\Delta A) + (\Delta A)^T P|_\infty < 1. \tag{3.12}$$

We have

$$\begin{aligned} |P(\Delta A) + (\Delta A)^T P|_\infty &\leq |P|_\infty|\Delta A|_\infty + |(\Delta A)^T|_\infty|P|_\infty \\ &= |P|_\infty|\Delta A|_\infty + |\Delta A|_1|P|_\infty \\ &\leq 2|P|_\infty \max\{|\Delta A|_\infty, |\Delta A|_1\} \\ &= 2|P|_\infty\alpha(\Delta A) < 1 \end{aligned}$$

where we have used (3.9). This completes the proof of the lemma. □

Next, we determine sufficient conditions for Schur stability.

Lemma 1b: If $[A^m, A^M]$ satisfies Assumption 1b, then $[A^m, A^M]$ is Schur stable.

Proof: As in the proof of Lemma 1a, we only need to show that for any $A \in [A^m, A^M]$, the matrix $A^T PA - P$ is negative definite, where $P = P^T$ is determined by (3.3). To accomplish this it suffices to show, similarly as in the proof of Lemma 1a, that

$$|(\Delta A)^T PA_0 + A_0^T P(\Delta A) + (\Delta A)^T P(\Delta A)|_\infty < 1 \tag{3.13}$$

where $\Delta A = A - A_0 = A - (1/2)(A^m + A^M)$ satisfies (3.8).

To show that (3.13) is true, we note that

$$\begin{aligned} &|(\Delta A)^T PA_0 + A_0^T P(\Delta A) + (\Delta A)^T P(\Delta A)|_\infty \\ &\quad\leq |\Delta A|_1|P|_\infty|A_0|_\infty + |A_0|_1|P|_\infty|\Delta A|_\infty \\ &\quad\quad+ |\Delta A|_1|P|_\infty|\Delta A|_\infty \\ &\quad= (|\Delta A|_1|A_0|_\infty + |\Delta A|_\infty|A_0|_1 \\ &\quad\quad+ |\Delta A|_1|\Delta A|_\infty)|P|_\infty \\ &\quad\leq (2\alpha(\Delta A)\cdot\alpha(A_0) + [\alpha(\Delta A)]^2)|P|_\infty \end{aligned} \tag{3.14}$$

where $\alpha(\cdot)$ is defined in (2.1).

Let $s = \alpha(\Delta A_0) \geq 0$. By (3.14), it suffices to show that

$$s^2 + 2\alpha(A_0)s < \frac{1}{|P|_\infty} \tag{3.15}$$

to prove that (3.13) is true. By (3.4) and (3.8) we have

$$s \leq \frac{1}{2}\alpha(A^M - A^m) < \left[(\alpha(A_0))^2 + \frac{1}{|P|_\infty}\right]^{1/2} - \alpha(A_0). \tag{3.16}$$

We note that the right-hand side of (3.16) is the largest root of the quadratic equation

$$\lambda^2 + 2\alpha(A_0)\lambda - \frac{1}{|P|_\infty} = 0 \tag{3.17}$$

where $\lambda \in R$.

Noting also that

$$s \geq 0 > -\alpha(A_0) - \left[(\alpha(A_0))^2 + \frac{1}{|P|_\infty}\right]^{1/2} \tag{3.18}$$

we conclude from (3.16) and (3.18) that s is larger or equal to the smallest root of (3.17) and smaller or equal to the largest root of (3.17). Therefore,

$$s^2 + 2\alpha(A_0)s - \frac{1}{|P|_\infty} < 0$$

and hence, (3.15) is true. This completes the proof of the lemma. □

Remarks 1: a) Previous results by the present authors [2] which provide sufficient conditions for the Hurwitz and Schur stability of interval matrices involve checks of the "corners" of hypercubes in R^{n^2} determined by interval matrices while the present results involve checkes on matrix norms, in particular, the norm determined by the function $\alpha(\cdot)$ defined in (2.1). Thus, the results in [2] and the present results are distinct and neither appears to imply the other.

b) One of the motivations for establishing the sufficient conditions given in Lemmas 1a and 1b is the possibility of establishing necessary and sufficient conditions for the Hurwitz and Schur stability of interval matrices. This will be accomplished in the next section. □

IV. Necessary and Sufficient Conditions

In this section we utilize the results of Section III to establish necessary and sufficient conditions for the Hurwitz and Schur stability of interval matrices.

Theorem 1a: An interval matrix $[A^m, A^M]$ is Hurwitz stable if and only if there are finitely many subinterval matrices $[A_i^m, A_i^M] \subset [A^m, A^M], 1 \le i \le k$, such that

$$[A^m, A^M] = \bigcup_{i=1}^{k} [A_i^m, A_i^M] \tag{4.1}$$

and for each $1 \le i \le k, [A_i^m, A_i^M]$ satisfies Assumption 1a. □

In the proof of Theorem 1a we will make use of the following necessary condition (which is also obviously a sufficient condition) for the Hurwitz stability of an interval matrix.

Lemma 2a: Assume that $[A_m, A^M]$ is Hurwitz stable. Then there exists a constant $r > 0$ such that for any subinterval $[A_0^m, A_0^M] \subset [A^m, A^M], [A_0^m, A_0^M]$ satisfies Assumption 1a, as long as $\alpha(A_0^M - A_0^m) < r$, where $\alpha(\cdot)$ is defined by (2.1).

Proof: By assumption, every $A \in [A^m, A^M]$ is Hurwitz stable. Therefore, there exists a positive definite matrix $P = P(A) = P^T$ which satisfies the matrix equation

$$PA + A^T P = -I \tag{4.2}$$

where $I \in R^{n \times n}$ is the identity matrix. Since $[A^m, A^M]$ is a compact set in R^{n^2}, and since every continuous function on a compact set assumes its minimal value, there exists a constant $r > 0$ such that

$$r \le \frac{1}{|P|_\infty} \tag{4.3}$$

for all $A \in [A^m, A^M]$ (recall that $P = P(A)$).

For any $[A_0^m, A_0^M] \subset [A^m, A^M]$ satisfying $\alpha(A_0^M - A_0^m) < r$, (4.3) implies that

$$\alpha(A_0^M - A_0^m) < \frac{1}{|P|_\infty} \tag{4.4}$$

where $P_0 = P(A_0)$ and $A_0 = (1/2)(A_0^m + A_0^M) \in [A^m, A^M]$.

The Hurwitz stability of A_0 along with (4.4) imply now that $[A_0^m, A_0^M]$ satisfies Assumption 1a. □

Using Lemmas 1a and 2a, we now prove Theorem 1a.

Proof of Theorem 1a: (*Sufficiency*) Assume that $[A_i^m, A_i^M]$ satisfies Assumption 1a for each $1 \le i \le k$. By Lemma 1a, $[A_i^m, A_i^M]$ is Hurwitz stable, $1 \le i \le k$, and thus, by (4.1), $[A^m, A^M]$ is Hurwitz stable.

(*Necessity*) By Lemma 2a, there exists a constant $r > 0$ such that for any subinterval $[A_0^m, A_0^M] \subset [A^m, A^M], [A_0^m, A_0^M]$ satisfies Assumption 1a, as long as $\alpha(A_0^M - A_0^m) < r$ where $\alpha(\cdot)$ is defined by (2.1).

Since $[A^m, A^M]$ is a hyperrectangle in R^{n^2}, we can subdivide it into a finite number of hyperrectangles $[A_i^m, A_i^M], 1 \le i \le k$, such that $\alpha(A_i^M - A_i^m) < r$ for all $1 \le i \le k$ (note that for any $A \in R^{n \times n}$, $\alpha(A) \le n \max_{1 \le i, j \le n}(|a_{ij}|)$). Therefore, by Lemma 2a, the subintervals $[A_i^m, A_i^M], 1 \le i \le k$, satisfy Assumption 1a.

This completes the proof of the theorem. □

To establish necessary and sufficient conditions for the Schur stability of an interval matrix, we proceed similarly as in the case of Hurwitz stability of such matrices.

Theorem 1b: An interval matrix $[A^m, A^M]$ is Schur stable if and only if there are finitely many subinterval matrices $[A_i^m, A_i^M] \subset [A^m, A^M], 1 \le i \le k$, such that

$$[A^m, A^M] = \bigcup_{i=1}^{k} [A_i^m, A_i^M] \tag{4.5}$$

and for each $1 \le i \le k$, $[A_i^m, A_i^M]$ satisfies Assumption 1b. □

In the proof of Theorem 1b, use is made of the following necessary condition (which is also obviously a sufficient condition) for the Schur stability of an interval matrix.

Lemma 2b: Assume that $[A^m, A^M]$ is Schur stable. Then there exists a constant $d > 0$ such that for any subinterval $[A_0^m, A_0^M] \subset [A^m, A^M], [A_0^m, A_0^M]$ satisfies Assumption 1b, as long as $\alpha(A_0^M - A_0^m) < d$, where $\alpha(\cdot)$ is defined in (2.1). □

The proofs of Lemma 2b and Theorem 1b proceed along similar lines as the proofs of Lemma 2a and Theorem 1a, respectively. In the interests of brevity, we omit the details.

Before considering specific cases, we note that Theorem 1a (theorem 1b) and Lemma 2a (Lemma 2b) enable us to ascertain the Hurwitz (Schur) stability of a given interval matrix by subdividing this interval into a sufficiently large number of subintervals which are sufficiently small, and then, by determining the Hurwitz (Schur) stability of each subinterval, using Lemma 1a (Lemma 1b). Lemma 2a (Lemma 2b) ensures that if the interval matrix under study is Hurwitz (Schur) stable, then we can always subdivide the interval into sufficiently many subintervals (with sufficiently small sizes) so that each subinterval satisfies Assumption 1a (Assumption 1b). These observations are the basis of an algorithm developed in Section VI.

V. Quadratic Lyapunov Functions

Suppose that an interval matrix $[A^m, A^M]$ is Hurwitz stable. A natural question which arises is whether there exists a positive definite quadratic Lyapunov function $v(x) = x^T P x, P = P^T$, such that for all $A \in [A^m, A^M]$ the time derivative of v along the solutions of

$$\dot{x} = Ax \tag{5.1}$$

given by

$$\dot{v}_{(5.1)}(x) = x^T(PA + A^T P)x \tag{5.2}$$

is negative definite. A similar question arises in connection with Schur stability of interval matrices (for linear, time-invariant discrete-time systems). Answers to these questions are negative in general (see [7]). The results in the present paper, however, provide additional understanding in this matter.

Definition 1: An interval matrix $[A^m, A^M]$ is said to be quadratically Hurwitz stable if there exists a positive definite matrix $P = P^T$ such that $PA + A^T P$ is negative definite for all $A \in [A^m, A^M]$. The interval matrix $[A^m, A^M]$ is said to be quadratically Schur stable if there exists a positive definite matrix $P = P^T$ such that $A^T PA - P$ is negative definite for all $A \in [A^m, A^M]$. □

An examination of the proofs indicates that in Lemmas 1a and 1b we actually established the following results.

Corollary 1: If $[A^m, A^M]$ satisfies Assumption 1a, then $[A^m, A^M]$ is quadratically Hurwitz stable. If $[A^m, A^M]$ satisfies Assumption 1b, then $[A^m, A^M]$ is quadratically Schur stable. □

From the definitions, we see that quadratic Hurwitz (Schur) stability implies Hurwitz (Schur) stability for interval matrices. As indicated earlier, the converse to the above is not true. In view of Theorems 1a and 1b, however, we can at least make the following connection between the concepts of Hurwitz (Schur) stability and quadratic Hurwitz (Schur) stability of interval matrices.

Theorem 2: An interval matrix $[A^m, A^M]$ is Hurwitz (Schur) stable if and only if there are a finite number of subintervals $[A_i^m, A_i^M], 1 \le i \le k$, such that

$$[A^m, A^M] = \bigcup_{i=1}^{k}[A_i^m, A_i^M] \tag{5.3}$$

and for each $1 \le i \le k$, $[A_i^m, A_i^M]$ is quadratically Hurwitz (Schur) stable.

Proof: Due to the similarities of proofs, we consider only the case of Hurwitz stability.

(*Sufficiency*) If for each $1 \le i \le k$, $[A_i^m, A_i^M]$ is quadratically Hurwitz stable, then $[A_i^m, A_i^M]$ is Hurwitz stable. Therefore, $[A^m, A^M] = \bigcup_{i=1}^{k}[A_i^m, A_i^M]$ is Hurwitz stable.

(*Necessity*) By Theorem 1a, there is a finite number of subintervals, $[A_i^m, A_i^M], 1 \le i \le k$, such that (5.3) holds and for each $1 \le i \le k$, $[A_i^m, A_i^M]$ satisfies Assumption 1a. Therefore, by Corollary 1, $[A_i^m, A_i^M]$ is quadratically Hurwitz stable, $1 \le i \le k$. □

VI. An Algorithm

In the present section we develop an algorithm which is based on Theorems 1a and 1b to test the Hurwitz and Schur stability of interval matrices. We demonstrate the applicability of our algorithm by means of two specific examples.

In the following algorithm, for any given interval matrix $[A^m, A^M]$, we first determine the Hurwitz stability of the matrix $B = (1/2)(A^m + A^M)$ by solving the Lyapunov equation $PB + B^TP = -I$, where $P = P^T$. If the solution is not unique or is not positive definite, the algorithm terminates with the result that $[A^m, A^M]$ is not Hurwitz stable. If the unique solution P is positive definite, then we verify if Assumption 1a is satisfied. If Assumption 1a is satisfied, then the algorithm terminates with the result that $[A^m, A^M]$ is Hurwitz stable. Otherwise, we divide the interval $[A^m, A^M]$ into two equal subintervals and repeat the above process for each subinterval. The algorithm continues unless each subinterval of $[A^m, A^M]$ is determined to be Hurwitz stable or at least one of the subintervals of $[A^m, A^M]$ is determined to be not Hurwitz stable, in the manner described above.

Problem: Given an interval matrix $[A^m, A^M]$ with $A^m = [a_{ij}^m]$ and $A^M = [a_{ij}^M]$, determine its Hurwitz stability properties, under the assumption that A^m and A^M are Hurwitz stable.

Algorithm

1) Initialization: $A_1^m = [a_{ij}^1] = A^m$, $A_1^M = [\overline{a}_{ij}^1] = A^M$ and $K_1 = \{1\}$.
2) Let $K = K_1$.
3) For every $k \in K$, compute $B_k = (1/2)(A_k^m + A_k^M) = [b_{ij}^k]_{n\times n}$ and solve for P_k from $P_kB_k + B_k^TP_k = -I$ with $P_k = P_k^T$.
4) If for every $k \in K$, P_k is the unique solution and is positive definite, go to Step 4. Otherwise, the interval matrix $[A^m, A^M]$ is not Hurwitz stable. Stop
5) For every $k \in K$, compute $C_k = A_k^M - A_k^m = [c_{ij}^k]_{n\times n}$, $\alpha_k = \max\{|C_k|_1, |C_k|_\infty\}$ and $\beta_k = (1/|P_k|_\infty)(P_k$ is computed in Step 2.).
6) If for every $k \in K$, $\alpha_k < \beta_k$, the interval matrix $[A^m, A^M]$ is Hurwitz stable. Stop. Otherwise, determine $K_0 = \{k \in K{:}\, \alpha_k \ge \beta_k\}$, and go to Step 6.
7) For every $k \in K_0$, find the maximal element c_{pq}^k of the matrix C_k and partition $[A_k^m, A_k^M]$ into two interval matrices $[D_k^m, D_k^M]$ and $[E_k^m, E_k^M]$, where $D_k^m = A_k^m$, $E_k^M = A_k^M$, $D_k^M = [d_{ij}^k]$, and $E_k^m = [e_{ij}^k]$ with $\overline{d}_{ij}^k = \begin{cases} b_{ij}^k \\ \overline{a}_{ij}^k \end{cases}$ and $e_{ij}^k = \begin{cases} b_{ij}^k, & \text{if } i = p \text{ and } j = q \\ a_{ij}^k, & \text{otherwise} \end{cases}$.
8) Relabel the set

$$\{[D_k^m, D_k^M], [E_k^m, E_k^M], k \in K_0\} \tag{6.1}$$

using

$$\{[A_k^m, A_k^M], k \in K_1\} \tag{6.2}$$

where $K_1 = \{1, \cdots, N\}$ and N is the number of interval matrices in (6.1) or (6.2).

Go back to Step 1. □

Remarks 2: a) The number of elements in K_0 is smaller than or equal to the number of elements in K of Step 5.

b) If in Steps 3 and 5 we replace "Hurwitz" by "Schur" and in Step 4 we replace $\beta_k = (1/|P_k|\infty)$ by $\beta_k = 2\{[\alpha(B_k)^2 + (1/|P|_\infty)]^{1/2} - \alpha(B_k)\}$, then the algorithm will determine Schur stability properties of $[A^m, A^M]$. □

The above algorithm has been implemented on a Convex C240 machine. We demonstrate the applicability of the algorithm by the following two examples.

Example 1: The 4×4 interval matrix $[A^m, A^M]$, where

$$A^m = \begin{bmatrix} -3 & 4 & 4 & -1 \\ -4 & -4 & -4 & 1 \\ -5 & 2 & -5 & -1 \\ -1 & 0 & 1 & -4 \end{bmatrix} \quad \text{and}$$

$$A^M = \begin{bmatrix} -2 & 5 & 6 & 1.5 \\ -3 & -3 & -3 & 2 \\ -4 & 3 & -4 & 0 \\ 0.1 & 1 & 2 & 2.5 \end{bmatrix}$$

has been determined to be Hurwitz stable by the above algorithm, requiring 19 cycles. Execution of the entire process required 1372.98 CPU seconds. Because of the efficiency of the algorithm (see Remark 2a), only 11,345 matrices (B_k) out of a maximum possible of $1 + 2 + \cdots + 2^{19} = 2^{20} - 1 = 1{,}048{,}575$ matrices (B_k) had to be checked (involving Steps 2 through 5) in the Algorithm.

Example 2: For the 4×4 interval matrix $[A^m, A^M]$, where

$$A^m = \begin{bmatrix} -8 & 4 & 4 & -6 \\ -5 & -6.9 & -4 & 1 \\ -6 & 2 & -8.7 & -1 \\ -3.4 & 0 & 4 & -4.9 \end{bmatrix} \quad \text{and}$$

$$A^M = \begin{bmatrix} -2 & 7.7 & 6.8 & -2 \\ -1 & -2 & -1 & 2.2 \\ -4 & 5.5 & -2 & 4 \\ 0 & 3 & 5.6 & -3 \end{bmatrix}$$

both A^m and A^M are Hurwitz stable. Using the above algorithm, we determine that $[A^m, A^M]$ is not Hurwitz stable. Indeed, in the cycle, we obtain ($k = 63$)

$$B_{63} = \begin{bmatrix} -6.5 & 5.85 & 5.4 & -5 \\ -4 & -5.675 & -2.5 & 1.6 \\ -5 & 3.75 & -3.675 & 2.75 \\ -1.7 & 1.5 & 4.8 & -3.95 \end{bmatrix} \in [A^m, A^M]$$

which is not Hurwitz stable (i.e., it has a positive eigenvalue, $\lambda \cong 0.1538$). In the present case, the entire process requires only 10.63 CPU seconds. (The coefficients in B_{63} are precise and do not involve any round-off.) □

VII. Concluding Remarks

In the present paper we first established a set of new sufficient conditions for the Hurwitz and Schur stability of interval matrices. We used these results to establish necessary and sufficient conditions for the Hurwitz and Schur stability of interval matrices. We related the above results to the existence of quadratic Lyapunov functions for linear time-invariant systems with interval-valued coefficient matrices. Using the above results, we developed an algorithm to determine the Hurwitz stability properties of interval matrices for the cases when the eigenvalues of all matrices belonging to the interval matrix under investigation have negative real parts, or at least one matrix belonging to the interval matrix has an eigenvalue with positive real part. By making obvious modifications, this algorithm determines also the Schur stability properties of interval matrices for the cases when the eigenvalues of all matrices belonging to the interval matrix under investigation have magnitudes less than one, or at least one matrix belonging to the interval matrix has an eigenvalue with magnitude greater than one. We demonstrated the applicability of our results by means of two specific examples.

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Steady-State Behavior in the Vibrational Control of a Class of Nonlinear Systems by AP-Forcing

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***Abstract*— Vibrational control is an open-loop control technique that uses zero mean parametric vibrations for shaping the response of a linear or nonlinear dynamical system. Several theoretical results are available, based on averaging techniques, assuring the possibility of modifying the equilibrium properties of a system. For nonlinear systems, computational difficulties arise and theoretical results cannot easily be applied. In this note, the stationary behavior of a class of nonlinear systems vibrationally controlled by AP-forcing is investigated. A practical formula linking the amplitude and the frequency of the vibration and the amplitude of the steady-state oscillation in the controlled variable is obtained. As test cases, the well-known Rayleigh equation, a catalytic reactor equation, and the phase locked loop equation are considered.**

I. Introduction

Vibrational control has been proposed as an effective control technique in recent years [1], [2], [3]. Such a method makes use of zero mean parametric excitation so that on-line measurements on the system are not necessary. If measurements are not available, traditional control methods (such as feedback and feed-forward techniques) fail, but vibrational control can be successfully used as an open-loop tool for achieving the control objectives. Several applications of vibrational control can be found in the literature to ensure stabilization of particle beams [4], plasma [5], lasers [6], and chemical reactors [7]–[10]. Some difficulties are present both in the theory and in practical applications of vibrational control when the controlled plants are nonlinear. Typically the amplitude of the vibration at the output is not negligible, hence a correct estimation of this amplitude is of practical value. In this note a simple formula is derived, linking the amplitude and the frequency of the vibrations and the amplitude of the steady-state oscillation in the controlled variable; three examples are included to illustrate the technique.

II. Main Result

Consider the dynamic nonlinear system controlled by AP-forcing [2] vibrations

$$x^{(n)} + f(x(t), \cdots, x^{(n-1)}) = (A/\epsilon)\sin(t/\epsilon + \alpha) \qquad (1)$$

where $x(t) \in R$, $x^{(i)} = d^i x(t)/dt^i$, $i = 1, \cdots, n$.

The matrix state equation, if $x(t) = x_1(t)$, is given by

$$\begin{bmatrix} dx_1/dt \\ \cdots \\ dx_{n-1}/dt \\ dx_n/dt \end{bmatrix} = \begin{bmatrix} x_2 \\ \cdots \\ x_n \\ -f(x_1, \cdots, x_n) \end{bmatrix} + \begin{bmatrix} 0 \\ \cdots \\ 0 \\ (A/\epsilon)\sin((t/\epsilon) + \alpha) \end{bmatrix}. \qquad (2)$$

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Comments on "Necessary and Sufficient Conditions for the Hurwitz and Schur Stability of Interval Matrices"[1]

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Abstract—**This note shows that Example 1 in the above paper[1] is not Hurwitz stable.**

Recently, Wang *et al.*[1] derived some new sufficient and necessary conditions for the Hurwitz and Schur stability of interval matrices. Based on these results, they developed an elegant algorithm to determine the Hurwitz and Schur stability properties of interval matrices. In this note, we show that their Example 1 is not Hurwitz stable.

Example 1: Wang *et al.*[1] claim that the interval matrix $[A^m \ A^M]$ where

$$A^m = \begin{bmatrix} -3 & 4 & 4 & -1 \\ -4 & -4 & -4 & 1 \\ -5 & 2 & -5 & -1 \\ -1 & 0 & 1 & -4 \end{bmatrix} \text{ and } A^M = \begin{bmatrix} -2 & 5 & 6 & 1.5 \\ -3 & -3 & -3 & 2 \\ -4 & 3 & -4 & 0 \\ 0.1 & 1 & 2 & 2.5 \end{bmatrix}$$

is Hurwitz stable. If we choose, however, a matrix in $[A^m \ A^M]$ as

$$A = \begin{bmatrix} -2.5 & 4.5 & 5 & 0.25 \\ -3.5 & -3.5 & -3.5 & 1.5 \\ -4.5 & 2.5 & -4.5 & -0.5 \\ -0.45 & 0.5 & 1.5 & 0.875 \end{bmatrix}$$

then it is seen that matrix A with the eigenvalues at 0.6696, -2.7173, and $-3.7887 \pm j6.7546$ is unstable. Therefore the conclusion of Wang *et al.*[1] is incorrect.

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Correction to "Necessary and Sufficient Conditions for the Hurwitz and Schur Stability of Interval Matrices"[1]

Kaining Wang, Anthony N. Michel, and Derong Liu

In the above paper,[1] Example 1, the last entry in matrix A^M should be -2.5 instead of 2.5. This is a typographical error and in no way does it alter the validity of the results.

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