Information Sciences 235 (2013) 36–44

Contents lists available at SciVerse ScienceDirect

Information Sciences

journal homepage: www.elsevier.com/locate/ins

Data-based stability analysis of a class of nonlinear discrete-time systems

Zhuo Wang, Derong Liu *

Department of Electrical and Computer Engineering, University of Illinois, Chicago, IL 60607-7053, USA

ARTICLE INFO

Article history:
Available online 9 February 2012

Keywords:
Data-based methods
Nonlinear discrete-time systems
Stability analysis

ABSTRACT

Nowadays, with modern sensor technologies, the inputs, outputs and states of the system can be accurately measured. Based on this fact, analyzing the properties of the system, which has unknown mathematical model, directly by using the measured data, has become feasible. In this paper, some data-based methods are proposed for state stability analysis of a class of nonlinear discrete-time systems. We also discuss the problem of finding the domain of attraction of the equilibrium point, using these data-based methods.

1. Introduction

In recent years, with the development of science and technology, in particular with the rapid development of information science and technology, many businesses and industries have undergone great changes, such as chemical industry, electric power engineering, electronics, mechanical engineering, metallurgy, transportation, and logistics business. While the scale of industrial enterprises is increasing, the production equipments and industrial processes are becoming more and more complex. As a consequence, the traditional methods to control the production processes as well as the equipments, which need to establish the mathematical models of the systems based on their physical and chemical mechanisms, have become infeasible. Due to the modern digital sensor technologies and their universal applications, such as information collection, storage, transmission, and processing technologies, as well as their continuous developments, these industrial enterprises generate a vast amount of digital data on a daily basis, which reflects production processes, production performance, management of material tracking, and equipment operations. How to effectively use these online and offline data to combine data mining, pattern recognition and computer technologies such as parallel simulation with control theory and systems engineering, has become a very important issue that needs to be solved for the manufacturing, transportation, and logistics businesses [18].

Data-based methods, such as data-based control, data-based decision-making, data-based fault diagnosis, and data-based scheduling, are still not fully developed yet, although there have been some studies in the past ten years [5,6,11,14,23,31,33].

In practice, engineers often encounter such a problem that the traditional theories and approaches, which have to analytically construct the mathematical model of the system, can hardly meet the design and control requirements of complex nonlinear systems. Fortunately, for nonlinear discrete-time systems, the system information is contained within the sampled data of input, output and state. Thus, it might be feasible to analyze or control the system directly by using these measured data. This idea creates an urgent need for the research and technical support of data-based methods. The research on data-based control, decision making, fault diagnosis and scheduling, is obviously an interdisciplinary and challenging topic. The most often studied areas related to the above fields are machine learning, data-mining, computational intelligence as well as system and control theory [42]. In each of the above related areas, there are several favorite topics. For machine learning, it prefers lazy learning, supervised learning, unsupervised learning and reinforcement learning [3,17,20,24,30]. For data-mining, people usually do research on database management, parallel processing and visualization [12,13,22,25,39]. The

* Corresponding author. Tel.: +1 312 355 4475; fax: +1 312 996 6465.
E-mail addresses: wangzhuo12300@gmail.com (Z. Wang), derong@uic.edu (D. Liu).

0020-0255/$ - see front matter © 2012 Elsevier Inc. All rights reserved.
doi:10.1016/j.ins.2012.01.046

most often considered fields of computational intelligence are evolutionary computation, fuzzy systems and neural networks [10,19,21,28,36,45,46]. For system and control theory, there are two general research directions. One is about control theory, such as PID control, optimal control, adaptive control, iterative learning control, H_∞ control, model-free control, and model predictive control [2,4,7,8,10,16,37,40,41,43,44]; the other is about system theory, such as studies of system properties (stability, observability and controllability, etc.), system structures, Lyapunov theory, optimality, and performance analysis [27–29,32,35,38]. Based on this background, the data-based methods have attracted more and more attention and will be widely applied to solve industrial and real-life problems, which are encountered in the above fields.

Since the stability is an important characteristic of the system, and its study is also a great topic in control theory, we will analyze the system stability in this paper. However, as mentioned above, for large scale complex nonlinear discrete-time systems, the most prominent feature is the vast amount of sampled data, which could not provide a clear and effective mathematical model that can be analyzed. Here, large scale refers to the system scale ranging from a few dozen variables or nodes to a few thousand. The complexity arises from heterogeneous information sources, multi-modal signals, high nonlinearity, strong interactions among variables or system states, involvement of human activities, and mixture of tasks [34]. The consequence directly caused by these properties is the widespread uncertainties of the system that prevent the problems being solved by using classical model-based methods, which need the explicit knowledge of the system dynamic models. Fortunately, we might solve this problem from another perspective, which means studying the stability only by using the measured state data. If this idea is feasible, it will have great practical significance in industrial engineering. In this paper, we will develop some data-based methods for analyzing the stability of a class of nonlinear discrete-time systems.

2. System description and basic assumptions

In this section, we will study a class of nonlinear discrete-time systems in the following form:

$$x(k+1) = f(x(k), u(k))(k \geqslant 0), \tag{1}$$

where $x(k) \in R^n$ and $u(k) \in R^p$ are the state and the input, respectively, and k is the time index. $f(x(k), u(k))$ is an unknown nonlinear discrete-time function, in which no randomness is involved in the development of future states, and all of its parameters are uniform throughout the time (which is a deterministic lumped parameter system).

Since we study the stability problem of industrial production processes, the amount of measured data will be large but not infinite, and the range of state variables will be limited. Thus, we suppose that all the system states stay in a closed bounded convex region $D \subset R^n$, which contains the origin $x = 0$ in it.

In practical terms, if a dynamical system has zero input and zero state, the newly generated state will naturally become zero and the dynamic behaviors of the system will stay the same from then on. Thus, we assume that the origin $x = 0$ is an equilibrium point and $f(0, 0) \equiv 0$. Since we only discuss the state stability in this paper, we set $u(k) \equiv 0$. For convenience, let $g(x(k)) = f(x(k), 0)$. It is clear that studying the state stability of system (1) is equivalent to studying the stability of the following system:

$$x(k+1) = g(x(k))(k \geqslant 0). \tag{2}$$

In summary, we have the following fundamental assumptions.

Assumption 1. For system (2), we assume that

1. all the states $x(k) \in D$ can be measured and stored, where D is a closed bounded convex region in R^n;
2. $g(x(k))$ is continuous with respect to $x(k)$, and $g(0) \equiv 0$;
3. the origin $x = 0$ is the only equilibrium point, and there exists no limit cycle in D.

We will demonstrate that, it is feasible to use the measured data in D to analyze the state stability of system (1) (or equivalently, the stability of system (2)) by using the proposed methods.

3. Stability analysis using data-based methods

Before introducing the data-based stability analysis methods, we first give the following definitions, which will be used in the discussions.

Definition 1.([1]). In elementary geometry, a polyhedron is a geometric object with flat sides, which exists in any general number of dimensions. A convex polyhedron is a special case of a polyhedron, having the additional property that it is also a convex set of points in the n-dimensional space R^n.

Definition 2.([26]). Domain of attraction is also called basin of attraction. It is the region in state space of all initial conditions that tend to a particular solution such as a limit cycle, a fixed point, or other attractors.

From Definition 2, an equilibrium point has a domain of attraction if it is asymptotically stable. For a nonlinear discrete-time system, it is meaningful to know the boundary of the domain of attraction, because there is a very positive conclusion that any state trajectory starting inside that domain will converge to this equilibrium point. However, such boundary is very difficult to find, especially for high order systems and large scale complex nonlinear systems [9,28]. According to this situation, we may find some substitutes to overcome this difficulty. If we can clearly identify the boundary of a sub-domain of the domain of attraction, which also contains the asymptotically stable equilibrium point, the problem will become much easier to solve. In engineering practice, this sub-domain can be used for stability analysis instead of the original domain of attraction.

Now, we provide the following theorems, which apply the above definitions and the measured data to analyze the stability of system (2). In this paper, $\|\cdot\|$ denotes the Euclidean norm (or, 2-norm), and $|\cdot|$ denotes the absolute value.

Theorem 1. *For system (2), assume that in D, $g(x(k))$ and $\frac{dg(x(k))}{dx(k)}$ are both continuous with respect to $x(k)$, and $\frac{d^2g(x(k))}{dx^2(k)}$ exists. Therefore, in a neighborhood of $x = 0$, there exists the following Taylor expansion:*

$$g(x(k)) = g(0) + \frac{dg(x(k))}{dx(k)}|_{x(k)=0}x(k) + R(x(k)),$$

where $R(x(k))$ is the remainder term. If

1. *in a neighborhood of the origin $x = 0$, arbitrarily select n linearly independent initial states $x_1(0), x_2(0), \ldots, x_n(0)$, which have the same norm $0 < r \ll 1$, the measured trajectories $x_1(k), x_2(k), \ldots, x_n(k)$ $(k = 0, 1, \ldots)$ satisfy*

$$\|[x_1(1), x_2(1), \ldots, x_n(1)][x_1(0), x_2(0), \ldots, x_n(0)]^{-1}\| \leqslant \alpha < 1;$$

2. *$\forall x(k) \in D$, the remainder term $R(x(k))$ satisfies*

$$\|R(x(k))\| \leqslant \|x(k)\|^2 (k \geqslant 0);$$

then, we can conclude that

1. *the origin $x = 0$ is asymptotically stable;*
2. *the open ball centered at the origin whose radius is $(1 - \alpha)$, is a part of the domain of attraction.*

Proof. For any state $x(k) \in D$ $(x(k) \neq 0)$, let

$$\begin{aligned} A_0 &= \left.\frac{dg(x(k))}{dx(k)}\right|_{x(k)=0}, \\ A(x(k)) &= A_0 + \frac{R(x(k))x^T(k)}{\|x(k)\|^2}. \end{aligned} \tag{3}$$

Since $g(0) \equiv 0$, the Taylor expansion can be rewritten as:

$$x(k+1) = A_0x(k) + R(x(k)) = A(x(k))x(k). \tag{4}$$

Because $g(x(k))$ and $\frac{dg(x(k))}{dx(k)}$ are both continuous with respect to $x(k)$, $\|R(x(k))\| \leqslant \|x(k)\|^2$, and $r \ll 1$, we can infer that in the neighborhood of $x = 0$, where the initial states $x_1(0), x_2(0), \ldots, x_n(0)$ stay, $g(x(k))$ satisfies

$$x_i(1) = g(x_i(0)) = A_0x_i(0) \quad (1 \leqslant i \leqslant n). \tag{5}$$

Therefore,

$$[x_1(1), x_2(1), \ldots, x_n(1)] = A_0[x_1(0), x_2(0), \ldots, x_n(0)].$$

Since $x_i(0)$ $(1 \leqslant i \leqslant n)$ are linearly independent, $[x_1(0), x_2(0), \ldots, x_n(0)]^{-1}$ exists. Then,

$$\begin{aligned} &A_0 = [x_1(1), x_2(1), \ldots, x_n(1)][x_1(0), x_2(0), \ldots, x_n(0)]^{-1} \\ &\quad \Rightarrow \|A_0\| = \|[x_1(1), x_2(1), \ldots, x_n(1)][x_1(0), x_2(0), \ldots, x_n(0)]^{-1}\|. \end{aligned} \tag{6}$$

From Condition 1, we can obtain

$$\|A_0\| \leqslant \alpha < 1. \tag{7}$$

By (3), (7) and Condition 2, for any non-zero state $x(k) \in D$, we can obtain

$$\|A(x(k))\| = \left\|A_0 + \frac{R(x(k))x^T(k)}{\|x(k)\|^2}\right\| \leqslant \|A_0\| + \frac{\|R(x(k))\| \cdot \|x(k)\|}{\|x(k)\|^2} \leqslant \|A_0\| + \frac{\|x(k)\|^2}{\|x(k)\|} \leqslant \alpha + \|x(k)\|, \tag{8}$$

which means $\|A(x(k))\| \leqslant \alpha + \|x(k)\|$. Obviously, if $\|x(k)\| < 1-\alpha$, then $\|A(x(k))\| < \alpha + (1-\alpha) = 1$. Define

$$S = \{x(k) | x(k) \in D, \|x(k)\| < 1-\alpha\}, \tag{9}$$

which is an open ball centered at $x = 0$ with radius $(1-\alpha)$. By (4), all the non-zero states in S will satisfy

$$\forall k \geqslant 0, \|x(k+1)\| = \|A(x(k))x(k)\| \leqslant \|A(x(k))\| \cdot \|x(k)\| < \|x(k)\|. \tag{10}$$

Since $g(0) \equiv 0$, $x = 0$ is the only equilibrium point and there exists no limit cycle in D, then by (10), we shall have

$$\forall x(k) \in S, \lim_{k\to\infty} \|x(k)\| = 0 \Rightarrow \lim_{k\to\infty} x(k) = 0.$$

Therefore, we can conclude that the origin $x = 0$ is asymptotically stable; and by Definition 2, S is a part of the domain of attraction. □

Remark 1. There are some aspects that merit our attention.

1. For Condition 1 of Theorem 1, the reason why we choose n linearly independent initial states is that, it is needed to give full consideration to all dimensions of the state space, to ensure that no counter example exists in the neighborhood of the origin. In practical engineering, to check whether this condition holds, it is suggested to test more than one group of n linearly independent initial states. Each group has at least one initial state not belonging to the other groups. If all these states satisfy this condition, we can consider that Condition 1 holds.
2. Gradually decrease the value of $\alpha \in (0, 1)$, until Condition 1 no longer holds. We can obtain the maximum $(1-\alpha)$ as the radius of the open ball, which is a part of the domain of attraction.
3. For Condition 2 of Theorem 1, there is a general expression, which is

$$\|R(x(k))\| \leqslant \|x(k)\|^{\gamma} (k \geqslant 0),$$

where $\gamma \geqslant 2$ is a positive integer. The Conclusion 2 will become that the open ball centered at the origin whose radius is $\sqrt[\gamma-1]{1-\alpha}$, is a part of the domain of attraction, while the rest of the theorem do not change. Then, (8) will lead to $\|A(x(k))\| \leqslant \alpha + \|x(k)\|^{\gamma-1}$. Obviously, if $\|x(k)\|^{\gamma-1} < 1-\alpha$, then $\|A(x(k))\| < \alpha + (1-\alpha) = 1$. Therefore, we redefine S as follows:

$$S = \{x(k) | x(k) \in D, \|x(k)\| < \sqrt[\gamma-1]{1-\alpha}\},$$

which is an open ball centered at $x = 0$ with radius $\sqrt[\gamma-1]{1-\alpha}$. As a result, we can still obtain (10), and by Definition 2, this new S is a part of the domain of attraction.

Now, we introduce the following second-order nonlinear system as an example for Theorem 1.

Example 1

$$x(k+1) = g(x(k)) = \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.5x_1(k) + 0.2x_2^2(k) \\ 0.5x_2(k) \end{bmatrix}. \tag{11}$$

Obviously, $x = 0$ is the only equilibrium point.

It can be seen that $g(x(k))$ and $\frac{dg(x(k))}{dx(k)}$ are both continuous with respect to $x(k)$, and $\frac{d^2g(x(k))}{dx^2(k)}$ exists. Therefore, we can obtain the following Taylor expansion:

$$g(x(k)) = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.2x_2^2(k) \\ 0 \end{bmatrix}. \tag{12}$$

The remainder term is $R(x(k)) = [0.2x_2^2(k), 0]^T$. As a result,

$$\|x(k)\|^2 - \|R(x(k))\| = x_1^2(k) + x_2^2(k) - 0.2x_2^2(k) = x_1^2(k) + 0.8x_2^2(k) \geqslant 0. \tag{13}$$

Then, $\|x(k)\|^2 \geqslant \|R(x(k))\|$, and Condition 2 is satisfied. Select two linearly independent initial states $x(0) = [-0.02, 0.02]^T$ and $y(0) = [-0.02, -0.02]^T$ for example, which have the same norm $0.0283 \ll 1$. By (11), we can obtain

$$x(1) = [-0.00992, 0.01]^T, y(1) = [-0.00992, -0.01]^T \Rightarrow \|[x(1), y(1)][x(0), y(0)]^{-1}\| = 0.5. \tag{14}$$

Let $\alpha = 0.5$, then Condition 1 is satisfied. Thus, we obtain $\left\| \frac{dg(x(k))}{dx(k)}|_{x(k)=0} \right\| \leqslant 0.5 < 1$. Therefore, we can conclude that $x = 0$ is asymptotically stable.

We then construct an open disk

$$S = \{x(k) | x(k) \in D, \|x(k)\| < 1-\alpha = 0.5\}, \tag{15}$$

which is illustrated in Fig. 1. By Theorem 1, S given in (15) is a part of the domain of attraction.

On the other hand, the mathematical expression of system (11) is given. We analytically study the stability of the equilibrium point $x = 0$ and the domain of attraction. From (3), we can obtain

$$A_0 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad A(x(k)) = \begin{bmatrix} 0.5 & 0.2x_2(k) \\ 0 & 0.5 \end{bmatrix}. \tag{16}$$

Therefore, $\|A_0\| = 0.5 < 1$, and we can conclude that the origin $x = 0$ is asymptotically stable. For $A(x(k))$, we have

$$\|A(x(k))\| = \sqrt{\lambda_{\max}[A^T(x(k))A(x(k))]}, \tag{17}$$

where $\lambda_{\max}[A^T(x(k))A(x(k))]$ is the largest eigenvalue of $[A^T(x(k))A(x(k))]$. Such that, by solving $\det[\lambda I - A^T(x(k))A(x(k))] = 0$, where I is the identity matrix, we can obtain

$$\lambda_{\max}[A^T(x(k))A(x(k))] = 0.5\left\{[0.5 + x_2^2(k)/25] + \sqrt{[0.5 + x_2^2(k)/25]^2 - 0.25}\right\}.$$

By setting $\sqrt{\lambda_{\max}[A^T(x(k))A(x(k))]} < 1$, we can obtain the real domain of attraction:

$$S_T = \{x(k) | x_1(k) \in R, -3.75 < x_2(k) < 3.75\}. \tag{18}$$

Obviously, S given in (15) is a part of the domain of attraction S_T.

Comparing the two methods, when analyzing the stability of the equilibrium point, they have the same conclusion; while the data-based method can only find a part of the domain of attraction.

In addition, we arbitrarily select some initial states in S, and simulate the trajectories starting from them with Matlab program. These initial points are (0.3, 0.3), (−0.2, 0.3), (−0.2, −0.25) and (0.3, 0). Their trajectories are illustrated in Fig. 1, which converge to the origin $x = 0$. Since S_T given in (18) is much larger than S, to deliver our results more clearly, we do not show it here.

In order to introduce the next theorem, we use

$$x_{(\beta,i)}(0) = [0, \cdots, 0, \beta, 0, \cdots, 0]^T$$

($i = 1, 2, \ldots, n$) to represent the vector whose ith element is $\beta > 0$ and all other elements are zeros. Similarly, let

$$x_{(-\beta,i)}(0) = [0, \cdots, 0, -\beta, 0, \cdots, 0]^T$$

represent the vector, which has the ith element as $(-\beta)$, while all other elements are zeros. Assume that $x_{(\beta,i)}(0), x_{(-\beta,i)}(0) \in D$ $(1 \leqslant i \leqslant n)$. Otherwise, we adjust the value of β until all of them are within D.

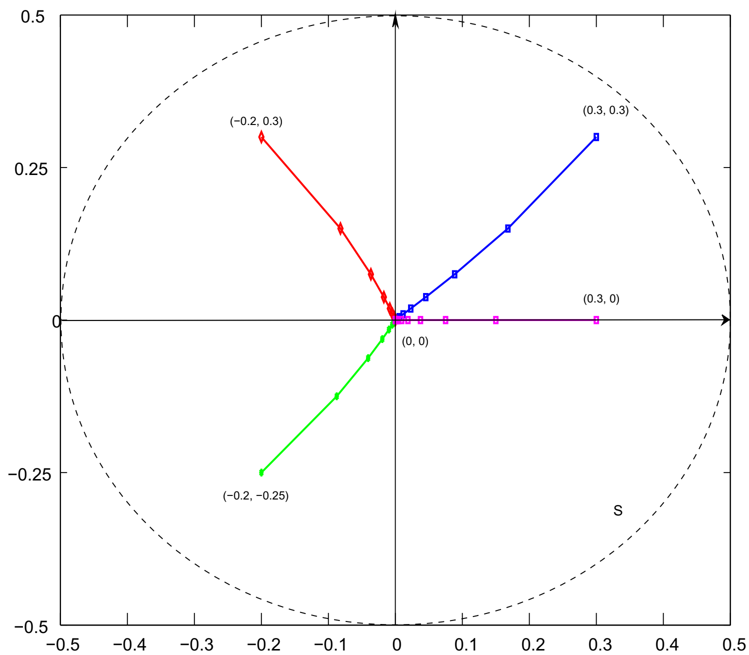

Fig. 1. The open disk S, which is a part of the domain of attraction, and the state trajectories starting from (0.3, 0.3), (−0.2, 0.3), (−0.2, −0.25) and (0.3, 0) for Example 1.

Theorem 2. *For system (2), if*

1. $\forall x(k),\ y(k) \in D$, *where* $x(k) \neq y(k)$ $(k \geqslant 0)$, *function g has the following properties:*
   (a) for any real number $0 \leqslant \alpha \leqslant 1$,

   $$|g_i[\alpha x(k) + (1-\alpha)y(k)]| \leqslant |\alpha g_i(x(k)) + (1-\alpha)g_i(y(k))|,$$

   where g_i represents the ith element of g $(i = 1, 2, \ldots, n)$;
   (b) $|x_i(k)| \leqslant |y_i(k)| \Rightarrow |g_i(x(k))| \leqslant |g_i(y(k))|$ $(i = 1, 2, \ldots, n)$;
2. there are $2n$ measured trajectories starting from $x_{(\beta,i)}(0)$ and $x_{(-\beta,i)}(0)$ $(i = 1, 2, \ldots, n)$, which satisfy the following inequalities:

   $$\forall k \geqslant 0, \quad \|x_{(\beta,i)}(k+1)\| < \|x_{(\beta,i)}(k)\|, \quad \|x_{(-\beta,i)}(k+1)\| < \|x_{(-\beta,i)}(k)\|;$$

then, we can conclude that

(1) the origin $x = 0$ is asymptotically stable;
(2) if we construct a closed convex polyhedron Ω, which has the $2n$ initial points $x_{(\beta,i)}(0)$, $x_{(-\beta,i)}(0)$, as its vertices, then Ω together with its interior is a part of the domain of attraction.

Proof. Since D is a closed bounded convex region and $x_{(\beta,i)}(0)$,$x_{(-\beta,i)}(0) \in D$ $(1 \leqslant i \leqslant n)$ by assumption, we shall have $\Omega \subset D$. Obviously, the origin is inside Ω. Among these $2n$ vertices of Ω, arbitrarily select two adjacent points, and denote them as $x_l(0)$ and $x_m(0)$, where $x_l(0)$, $x_m(0) \in \{x_{(\beta,i)}(0),\ x_{(-\beta,i)}(0),\ i = 1, 2, \ldots, n\}$. For any real number $0 \leqslant \alpha \leqslant 1$, there exists a point in D:

$$x_\alpha(0) = \alpha x_l(0) + (1-\alpha)x_m(0), \tag{19}$$

which is on the connection line between $x_l(0)$ and $x_m(0)$. Therefore, it is on an edge of Ω, and the jth element of $x_\alpha(0)$ is

$$x_{\alpha,j}(0) = \alpha x_{l,j}(0) + (1-\alpha)x_{m,j}(0)(1 \leqslant j \leqslant n), \tag{20}$$

where $x_{l,j}(0)$ and $x_{m,j}(0)$ are the jth elements of $x_l(0)$ and $x_m(0)$, respectively. Let $x_\alpha(k)$ $(k \geqslant 0)$ represent the trajectory starting from $x_\alpha(0)$. By (19), (20) and Condition 1(a), we can obtain

$$|x_{\alpha,j}(1)| = |g_j(x_\alpha(0))| = |g_j[\alpha x_l(0) + (1-\alpha)x_m(0)]| \leqslant |\alpha g_j(x_l(0)) + (1-\alpha)g_j(x_m(0))| = |\alpha x_{l,j}(1) + (1-\alpha)x_{m,j}(1)|. \tag{21}$$

Since D is a closed bounded convex region, $x_\alpha(k)$ and $[\alpha x_l(k) + (1-\alpha)x_m(k)]$ $(k \geqslant 0)$ will always stay in D [15]. Then by (21), Conditions 1(a) and 1(b), we shall have

$$|x_{\alpha,j}(2)| = |g_j(x_\alpha(1))| \leqslant |g_j[\alpha x_l(1) + (1-\alpha)x_m(1)]| \leqslant |\alpha g_j(x_l(1)) + (1-\alpha)g_j(x_m(1))| = |\alpha x_{l,j}(2) + (1-\alpha)x_{m,j}(2)|. \tag{22}$$

That is to say,

$$|x_{\alpha,j}(2)| \leqslant |\alpha x_{l,j}(2) + (1-\alpha)x_{m,j}(2)|.$$

By repeating the above process, we can obtain

$$\begin{aligned} |x_{\alpha,j}(k)| \leqslant |\alpha x_{l,j}(k) + (1-\alpha)x_{m,j}(k)|(1 \leqslant j \leqslant n, k \geqslant 0) \Rightarrow \sum_{j=1}^{n}|x_{\alpha,j}(k)|^2 \leqslant \sum_{j=1}^{n}|\alpha x_{l,j}(k) + (1-\alpha)x_{m,j}(k)|^2 \Rightarrow \|x_\alpha(k)\| \\ \leqslant \|\alpha x_l(k) + (1-\alpha)x_m(k)\| \leqslant \alpha\|x_l(k)\| + (1-\alpha)\|x_m(k)\|. \end{aligned} \tag{23}$$

From Condition 2, since $x_l(0)$, $x_m(0) \in \{x_{(\beta,i)}(0), x_{(-\beta,i)}(0),\ i = 1, 2, \ldots, n\}$, then their trajectories satisfy

$$\forall k \geqslant 0, \quad \|x_l(k+1)\| < \|x_l(k)\|, \quad \|x_m(k+1)\| < \|x_m(k)\|. \tag{24}$$

Since it is assumed that $x = 0$ is the only equilibrium point and there is no limit cycle in D, by (23) and (24), we have

$$\lim_{k\to\infty}\|x_l(k)\| = 0, \lim_{k\to\infty}\|x_m(k)\| = 0 \Rightarrow \lim_{k\to\infty}\|x_\alpha(k)\| = 0 \Rightarrow \lim_{k\to\infty}x_\alpha(k) = 0. \tag{25}$$

Similarly, since any point on the surface or within Ω can be expressed as a convex combination [15] of some points in $\{x_{(\beta,i)}(0), x_{(-\beta,i)}(0), 1 \leqslant i \leqslant n\}$, we can conclude that the state trajectory starting from any point on the surface or within Ω will converge to the origin $x = 0$.

Therefore, $x = 0$ is asymptotically stable; and by Definition 2, the closed convex polyhedron Ω together with its interior is a part of the domain of attraction.

To demonstrate Theorem 2, we introduce the following second-order system as an example. □

Example 2

$$x(k+1) = g(x(k)) = \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0.5x_1^2(k) + 0.5x_2^2(k) \\ 0.5x_2(k) \end{bmatrix}. \tag{26}$$

Obviously, the origin $x = 0$ is the only equilibrium point.

Arbitrarily select a real number $0 \leqslant \alpha \leqslant 1$, then for any $x(k), y(k) \in R^2$, we can obtain

$$\begin{aligned} g_1[\alpha x(k) + (1-\alpha)y(k)] &= 0.5\left\{[\alpha x_1(k) + (1-\alpha)y_1(k)]^2 + [\alpha x_2(k) + (1-\alpha)y_2(k)]^2\right\} \\ &= 0.5\{\alpha[x_1^2(k) + x_2^2(k)] + (1-\alpha)[y_1^2(k) + y_2^2(k)] - \alpha(1-\alpha)[x_1^2(k) + x_2^2(k)] - \alpha(1-\alpha)[y_1^2(k) + y_2^2(k)] \\ &\quad + 2\alpha(1-\alpha)[x_1(k)y_1(k) + x_2(k)y_2(k)]\} \\ &= \alpha g_1(x(k)) + (1-\alpha)g_1(y(k)) - 0.5\alpha(1-\alpha)\left\{[x_1(k) - y_1(k)]^2 + [x_2(k) - y_2(k)]^2\right\} \\ &\leqslant \alpha g_1(x(k)) + (1-\alpha)g_1(y(k)). \end{aligned} \tag{27}$$

Since $g_1[\alpha x(k) + (1-\alpha)y(k)]$ and $\alpha\, g_1(x(k)) + (1-\alpha)g_1(y(k))$ are both nonnegative, we shall have

$$|g_1[\alpha x(k) + (1-\alpha)y(k)]| \leqslant |\alpha g_1(x(k)) + (1-\alpha)g_1(y(k))|. \tag{28}$$

On the other hand,

$$\begin{aligned} g_2[\alpha x(k) + (1-\alpha)y(k)] = 0.5[\alpha x_2(k) + (1-\alpha)y_2(k)] = \alpha g_2(x(k)) + (1-\alpha)g_2(y(k)) \Rightarrow |g_2[\alpha x(k) + (1-\alpha)y(k)]| \\ = |\alpha g_2(x(k)) + (1-\alpha)g_2(y(k))|. \end{aligned} \tag{29}$$

From (27)–(29), we can infer that $g(x(k))$ given by (26) satisfies Condition 1(a). It can be seen that if $|x_j(k)| \leqslant |y_j(k)|$ then $|g_j(x(k))| \leqslant |g_j(y(k))|$ $(j = 1, 2)$. Thus, Condition 1(b) is also satisfied.

Choose $\beta = 1$, then the $2n$ initial points in Condition 2 are (1,0), (−1,0), (0,1) and (0,−1). By using Matlab program, we can simulate the trajectories starting from them, which are illustrated in Fig. 2. We can see that, the norms of these states are monotonically decreasing, and all of their trajectories converge to $x = 0$. Therefore, Condition 2 is satisfied. According to Theorem 2, $x = 0$ is asymptotically stable. Construct a quadrangle Ω, using (1, 0), (−1, 0), (0, 1), and (0, −1) as its vertices. Obviously, Ω is closed and convex. By Theorem 2, Ω together with its interior is a part of the domain of attraction.

Next, we use the analytical method to study the stability of the equilibrium point $x = 0$ and the domain of attraction. Define

$$A(k) = \begin{bmatrix} 0.5x_1(k) & 0.5x_2(k) \\ 0 & 0.5 \end{bmatrix}. \tag{30}$$

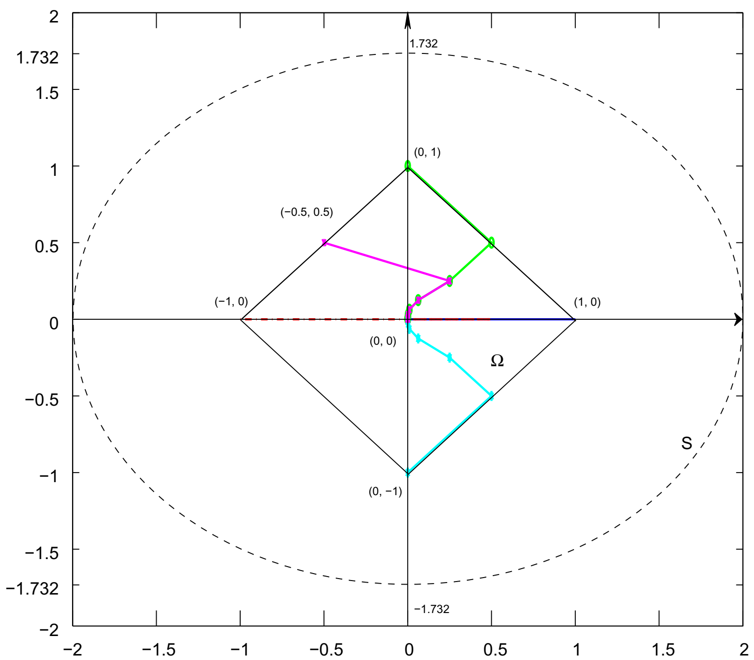

Fig. 2. The domain of attraction S, the quadrangle Ω and the state trajectories starting from (1, 0), (−1, 0), (0, 1), (0, −1), and (−0.5, 0.5) for Example 2.

Then, (26) can be rewritten as $x(k+1) = A(k)x(k)$. Since there is only one equilibrium point, the domain in which all the states satisfy $\|A(k)\| < 1$ will be the domain of attraction.

By solving $\det[\lambda I - A^T(k)A(k)] = 0$, where I is the identity matrix, we can obtain

$$\lambda_{\max}(A^T(k)A(k)) = \frac{1}{8}[x_1^2(k) + x_2^2(k) + 1] + \frac{1}{8}\sqrt{[x_1^2(k) + x_2^2(k) + 1]^2 - 4x_1^2(k)}.$$

Let $\sqrt{\lambda_{\max}(A^T(k)A(k))} < 1$, then the domain of attraction is

$$S = \left\{x(k) | \frac{x_1^2(k)}{4} + \frac{x_2^2(k)}{3} < 1\right\}, \tag{31}$$

which is illustrated in Fig. 2. Since $g(0) \equiv 0$, and in S we have $\|A(k)\| < 1$, then

$$\forall x(0) \in S, \lim_{k\to\infty} x(k) = 0.$$

Therefore, $x = 0$ is asymptotically stable. We can see that, Ω together with its interior is a part of the domain of attraction S.

Comparing the two methods, when analyzing the stability of the equilibrium point, they have the same conclusion; while the data-based method can only find part of the domain of attraction.

In addition, we arbitrarily select another initial point $(-0.5, 0.5) \in \Omega$. It can be seen that its trajectory also converges to $x = 0$. S, Ω and all of these trajectories are illustrated in Fig. 2.

4. Conclusions and future work

In this paper, we develop some data-based methods to solve the problem of stability analysis of a class of nonlinear discrete-time systems, without knowing the explicit mathematical expression of the system dynamic models. The advantage of Theorems 1 and 2 is that, one can determine whether the equilibrium point is asymptotically stable, just by measuring a group of state trajectories and doing some simple analysis on them, while setting the input to zero. Another good point is that, we can construct a part of the domain of attraction by using these measured data. We can then directly and positively conclude that, a state trajectory will converge to the equilibrium point, if it starts within that domain. This idea makes the problem of stability analysis much easier to solve, especially for the case of large-scale complex nonlinear systems. The theorems given in this paper are for local stability, but not for global stability. The latter one requires more conditions, which are not easily satisfied.

Our research gives a new perspective on analyzing the characteristics of nonlinear discrete-time systems. The future work of the data-based methods may be on some other topics, such as analyzing the controllability and the observability of the system, and building models of stochastic parameter systems.

References

[1] I. Agricola, T. Friedrich, Elementary Geometry, American Mathematical Society, 2008.
[2] B.D.O. Anderson, J.B. Moore, Optimal Control: Linear Quadratic Methods, Prentice-Hall, London, UK, 1989.
[3] C.M. Bishop, Pattern Recognition and Machine Learning, Springer, New York, 2006.
[4] T.Y. Chai, Challenges of optimal control for plant-wide production processes in terms of control and optimization theories, ACTA Automatica Sinica 35 (2009) 641–649.
[5] S. Chaudhuri, U. Dayal, V. Ganti, Database technology for decision support systems, Computer 34 (2001) 48–55.
[6] D.Z. Cheng, H.S. Qi, Z.Q. Li, Model construction of boolean network via observed data, IEEE Transactions on Neural Networks 22 (2011) 525–536.
[7] R.H. Chi, Z.S. Hou, Dual-stage optimal iterative learning control for nonlinear non-affine discrete-time systems, ACTA Automatica Sinica 33 (2007) 1061–1065.
[8] R. Findeisen, Assessment and Future Directions of Nonlinear Model Predictive Control, Springer, New York, 2007.
[9] H. Flashner, R.S. Guttalu, A computational approach for studying domains of attraction for non-linear systems, International Journal of Non-Linear Mechanics 23 (1988) 279–295.
[10] S.S. Ge, C.C. Hang, T.H. Lee, T. Zhang, Stable Adaptive Neural Network Control, Kluwer Academic, Boston, 2002.
[11] G.O. Guardabassi, S.M. Savaresi, Virtual reference direct design method: an off-line approach to data-based control system design, IEEE Transactions on Automatic Control 45 (2000) 954–959.
[12] J.W. Han, M. Kamber, Data Mining: Concepts and Techniques, Morgan Kaufman, San Francisco, CA, 2006.
[13] D.J. Hand, H. Mannila, P. Smyth, Principles of Data Mining, MIT Press, Cambridge, MA, 2000.
[14] K. Hinnen, M. Verhaegen, N. Doelman, A data-driven H_2-optimal control approach for adaptive optics, IEEE Transactions on Control Systems Technology 16 (2008) 381–395.
[15] J. Hiriart-Urruty, C. Lemarechal, Convex Analysis and Minimization Algorithms, Springer-Verlag, New York, 1996.
[16] H. Hjalmarsson, Iterative feedback tuning – an overview, International Journal of Adaptive Control and Signal Processing 16 (2002) 373–395.
[17] D.E. Holmes, L.C. Jain, Innovations in Machine Learning: Theory and Applications, Springer, New York, 2006.
[18] Z.S. Hou, J.X. Xu, On data-driven control theory: The state of the art and perspective, ACTA Automatica Sinica 35 (2009) 650–667.
[19] J. Jantzen, Foundations of Fuzzy Control, Wiley, New York, 2007.
[20] Y.C. Jin, Multi-Objective Machine Learning, Springer, Berlin, 2006.
[21] K.A.D. Jong, Evolutionary Computation: A Unified Approach, MIT Press, Cambridge, MA, 2006.
[22] D.A. Keim, Information visualization and visual data mining, IEEE Transactions on Visualization and Computer Graphics 8 (2002) 1–8.
[23] D. Kim, C.E. Rhee, S. Ha, Combined data-driven and event-driven scheduling technique for fast distributed cosimulation, IEEE Transactions on Very Large Scale Integration (VLSI) Systems 10 (2002) 672–678.
[24] M. Kobayashi, Y. Konishi, H. Ishigaki, A lazy learning control method using support vector regression, International Journal of Innovative Computing, Information and Control 3 (2007) 1511–1523.
[25] I. Kononenko, M. Kukar, Machine Learning and Data Mining: Introduction to Principles and Algorithms, Horwood Publishing, Chichester, 2007.

[26] S.M. Lavalle, Planning Algorithms, Cambridge University Press, New York, 2006.
[27] E. Lee, W.S. Lu, N. Wu, A Lyapunov theory for linear time-delay systems, IEEE Transactions on Automatic Control 31 (2003) 259.
[28] D.H. Lee, J.B. Park, Y.H. Joo, A fuzzy Lyapunov function approach to estimating the domain of attraction for continuous-time Takagi–Sugeno fuzzy systems, Information Sciences 185 (1) (2012) 230–248.
[29] S. Lee, Fairness, stability and optimality of adaptive multiagent systems: interaction through resource sharing, IEEE Transactions on Automation Science and Engineering 7 (2010) 427–439.
[30] F.L. Lewis, D. Vrabie, Reinforcement learning and adaptive dynamic programming for feedback control, IEEE Circuits and Systems Magazine 9 (2009) 32–50.
[31] I. Markovsky, P. Rapisarda, Data-driven simulation and control, International Journal of Control 81 (2008) 1946–1959.
[32] K.S. Narendra, A.M. Annaswamy, Stable Adaptive Systems, Prentice-Hall, London, 1989.
[33] P.F. Odgaard, B. Lin, S.B. Jorgensen, Observer and data-driven-model-based fault detection in power plant coal mills, IEEE Transactions on Energy Conversion 23 (2008) 659–668.
[34] F. Previdi, T. Schauer, S.M. Savaresi, K.J. Hunt, Data-driven control design for neuroprotheses: a virtual reference feedback tuning (VRFT) approach, IEEE Transactions on Control Systems Technology 12 (2004) 176–182.
[35] D.A. Reed, K.A. Shields, W.H. Scullin, L.F. Tavera, C.L. Elford, Virtual reality and parallel systems performance analysis, Computer 28 (1995) 57.
[36] B. Rezaee, F.M.H. Zarandi, Data-driven fuzzy modeling for Takagi–Sugeno–Kang fuzzy system, Information Sciences 180 (2) (2010) 241–255.
[37] F. Rezayat, On the use of an SPSA-based model-free controller in quality improvement, Automatica 31 (1995) 913–915.
[38] A. Sabanovic, L. Fridman, S.K. Spurgeon, Variable Structure Systems: From Principles to Implementation, IET Control Engineering Series 66, The Institution of Engineering and Technology, London, 2004.
[39] E.L. Silva, E.C. Finardi, Parallel processing applied to the planning of hydrothermal systems, IEEE Transactions on Parallel and Distributed Systems 14 (2003) 721–729.
[40] J.C. Spall, J.A. Cristion, Model-free control of nonlinear stochastic systems with discrete-time measurements, IEEE Transactions on Automatic Control 43 (1998) 1198–1210.
[41] K.K. Tan, Q.G. Wang, C.C. Hang, Advances in PID Control, Springer, London, UK, 1999.
[42] J.X. Xu, Z.S. Hou, Notes on data-driven system approaches, ACTA Automatica Sinica 35 (2009) 668–675.
[43] J.X. Xu, Z.S. Hou, On learning control: the state of the art and perspective, ACTA Automatica Sinica 31 (2005) 943–955.
[44] J.X. Xu, Y. Tan, Linear and Nonlinear Iterative Learning Control, Springer, Berlin, 2003.
[45] S. Yilmaz, Y. Oysal, Fuzzy wavelet neural network models for prediction and identification of dynamical systems, IEEE Transactions on Neural Networks 21 (2010) 1599–1609.
[46] L.A. Zadeh, Fuzzy Sets, Fuzzy Logic, and Fuzzy Systems, World Scientific, New York, 1996.