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Consensus protocol for multi-agent continuous systems*

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Based on the algebraic graph theory, the networked multi-agent continuous systems are investigated. Firstly, the digraph (directed graph) represents the topology of a networked system, and then a consensus convergence criterion of system is proposed. Secondly, the issue of stability of multi-agent systems and the consensus convergence problem of information states are all analysed. Furthermore, the consensus equilibrium point of system is proved to be global and asymptotically reach the convex combination of initial states. Finally, two examples are taken to show the effectiveness of the results obtained in this paper.

Keywords: multi-agent systems, consensus protocol, graph Laplacian, asymptotically stable
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1. Introduction

For many decades, scientists and biologists from diverse disciplines which include animal motion behaviour, physics, biophysics, social science, and computer science have been fascinated by the emergence of flocking, swarming, and schooling in groups of agents with local interactions.[1−6] In 1995, Vicsek *et al*[3] proposed a simple model for a system of several autonomous agents moving in the plane at the same speed but with different headings and then Jadbabaie *et al*[7] mathematically analysed that all agents will eventually move in a common direction of the Vicsek model. After that, many theoretical results and practical applications about multi-agent systems have been reported.[7−14]

The consensus problem originated from the field in management science and statistics in the 1970s.[15] About two decades later, the ideal of consensus theory by DeGroot was applied to the idea of statistical consensus in aggregation of information.[16] In networked multi-agent systems, the information flow and the interaction among agents play an important role in the group coordination. A critical problem for cooperative control is to design appropriate protocols and algorithms such that the group of agents can reach consensus on the shared information.[17] Consensus means reaching an agreement regarding a certain extent of interest that depends on the state of all agents. A consensus algorithm (or protocol) is an interaction rule that specifies the information exchange between an agent and all of its neighbours on the network.[18] The main relevant problems about consensus of agents are concerned with the synchronization of coupled oscillators, flocking and swarming, formation control, and rendezvous problems etc.[19−23]

Some useful tools for the analysis of multi-agent systems are matrix theory, algebraic graph theory, and control theory. The relationship between the graph theory and the control theory could be demonstrated by the graph Laplacian. In this paper, we focus our attention on a kind of linear time-invariant (LTI) system in order to explain the role of the graph in the system behaviour. We aim to explore the necessary and sufficient conditions for the stability of an interconnected agent system. It is also shown how to shape the information flow in terms of the graph Laplacian in the paper. The main results of the paper lie in the stability analysis of the multi-agent systems and the final consensus equilibrium point of the systems.

The rest of this paper is organized as follows. In Section 2, the mechanism of networked multi-agent systems is put forward, and some background of alge-

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braic graph theory and matrix theory are introduced. Analysed in Section 3 are the stability conditions and the performance for autonomous agent systems. Two examples are presented in Section 4, and the conclusion drawn from the present study is given in Section 5.

2. Consensus problem formulation and definitions

In this section, we introduce some basic concepts of consensus and notations of algebraic graph theory in multi-agent systems. Compared with conventional physics problems, one of typical technical challenges for this multi-agent consensus problem is how to analyse the effects of information flow topology among the agents. Let $\Sigma = \{\Sigma_i : i = 1,\ 2,\ \ldots,\ n\}$ represent a set of cooperative agents with the total number n. Assume that the communications among these agents are directed. So we define $G = (v,\ \varepsilon)$ as a directed graph where the n nodes represent n agents labelled as $\Sigma_1,\ \Sigma_2,\ \ldots,\ \Sigma_n$. Figure 1 is a network of integrator agents in which agent j receives the information from its neighbour agent i.

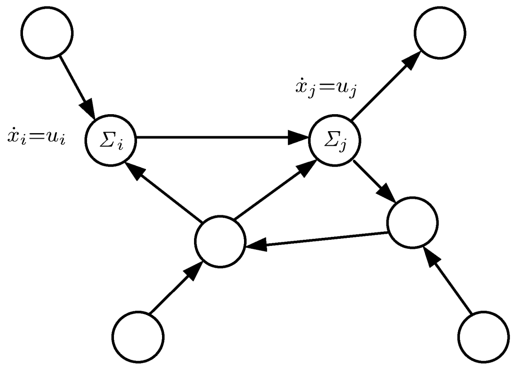

Fig.1. The topology of multi-agent systems.

2.1. Multi-agent systems and the concept of consensus

The information states with agent dynamics are given by

$$\dot{x}_i = u_i, i = 1, 2, \ldots, n, \qquad (1)$$

where $x_i \in R^n$ denotes the information state of the i^{th} agent and $u_i \in R^n$ is the control input. And the consensus algorithm to reach an agreement with respect to the states of n integrator agents (1) can be expressed as an n^{th}-order linear system on a graph, which was proposed in Refs.[7, 8, 11]

$$u_i = -\sum\nolimits_{j \in N_i} a_{ij}(x_i(t) - x_j(t)), x_i(0) \in R, \qquad (2)$$

where a_{ij} is the (i, j) entry of the adjacency matrix of the associated communication graph at time t, and N_i represents the set of agents whose information is available from agent Σ_i at time t. The control u_i drives x_i to the average position of its neighbours.

By applying algorithm (2), we can rewrite expression (1) into

$$\dot{x}(t) = -Lx(t), \qquad (3)$$

where $x = [x_1, \ldots, x_n]^{\text{T}}$ denotes the aggregated state vector of the multi-agent systems, and $L = [l_{ij}] \in R^{n\times n}$ is the graph Laplacian of the network.

Let $x_i \in R$ denote the value of node v_i. We refer to $G_x = (G, x)$ with $x = [x_1, \ldots, x_n]^{\text{T}}$ as a network (or algebraic graph) with value $x \in R^n$ and topology (or information flow) G. It is said that node v_i and node v_j agree in a network if $x_i = x_j$. We say that the nodes in a network have reached a consensus if $x_i = x_j$ for all $i, j \in \Sigma, i \neq j$. Whenever the states of a network are all in agreement, the common value of all nodes is called the group decision value. Based on system (3), we can obtain the following definition of the consensus algorithm.

Definition 1[10] The set of agents Σ is said to be in consensus, if $\|x_i - x_j\| = 0$ for each $(i, j) : i, j = 1,\ 2,\ \ldots,\ n$ as $t \geq t_0$. The set of agents Σ is said to asymptotically reach global consensus if for any $x_i(0) : i = 1,\ 2,\ \ldots,\ n$, $\|x_i - x_j\| \to 0$ as t tends to infinity for each $(i, j) : i, j = 1,\ 2,\ \ldots,\ n$. The set of agents Σ is said to be global consensus reachable if there exists an information update strategy (protocol) for each $x_i : i = 1,\ 2,\ \ldots,\ n$ that achieves global consensus asymptotically for Σ.

2.2. Algebraic graph theory

The interaction topology of a network of agents is represented by a directed graph $G = (v, \varepsilon)$ with the set of nodes $v = \{1,\ 2,\ \ldots,\ n\}$ and edges $\varepsilon = v \times v$. Each edge is denoted by $e_{ij} = (v_i,\ v_j)$. We refer to the v_i and v_j as the tail and head of the edge $(v_i,\ v_j)$. The neighbours of agent i are denoted by $\sum_i = \{j \in v : (i, , j) \in \varepsilon\}$.

Definition 2 The adjacency matrix $A = [a_{ij}] \in R^{n\times n}$ of a directed graph with node set $N = \{1, \ldots, n\}$ is defined as follows: a_{ij} is a positive weight if $(i,\ j) \in \varepsilon$, while $a_{ij} = 0$ if $(i,\ j) \notin \varepsilon$. Note that all graphs are weighted. If the weights are not relevant, then a_{ij} is set to be equal to 1 for all $(i,\ j) \in \varepsilon$.

Definition 3 Let $G = (v, \varepsilon)$ be a weighted directed graph (or digraph) with n nodes. The in-degree and out-degree of node v_i are, respectively, defined as $\deg_{\rm in}(v_i) = \sum_{j=1}^{n} a_{ji}$ and $\deg_{\rm out}(v_i) = \sum_{j=1}^{n} a_{ij}$. For a graph with $0-1$ adjacency elements, $\deg_{\rm out}(v_i) = |N_i|$. The degree matrix of the digraph G is a diagonal matrix $D = [D_{ij}]$ where $D_{ij} = 0$ for all $i \neq j$ and $D_{ij} = \deg_{\rm out}(v_i)$. The graph Laplacian associated with the digraph G is then defined as

$$L(G) = D - A. \tag{4}$$

3. Consensus protocol and main results

3.1. Stability analysis

A digraph is called being strongly connected if there is a directed path from one node to another node. A directed tree is a digraph, where every node, except the root, has exactly one parent. A directed spanning tree of a digraph is a directed tree formed by graph edges that connect all the nodes of graph.

Lemma 1 If the topology associated digraph is strongly connected, then the Laplacian matrix L has the following properties: 1) its eigenvalues have nonnegative real-parts; 2) the Kernel$\{-L\}$ = span$\{1\}$, where $1 = [1, \ldots, 1]^{\rm T} \in R^n$, implying that$\alpha = 1$ is an eigenvector corresponding to zero eigenvalue, i.e. $L\alpha = 0$, since $\sum_{j=1}^{N} L_{ij} = 0$; 3) the Rank$(L) = n-1$.

Based on Lemma 1, all row-sums of graph Laplacian matrix are zero. Therefore, L always has a zero eigenvalue, i.e. $\lambda_1 = 0$. This zero eigenvalue corresponds to the eigenvector $\alpha = [1, 1, \ldots, 1]^{\rm T}$ because α is in the null-space of L ($L\alpha = 0$). In other words, an equilibrium of system (3) is a state in the form $x^* = \alpha\,[1, \ldots, 1]^{\rm T}$ and exponentially stable, which implies that $\alpha = {\rm span}\,\{1\}$ is contained in the kernel of L. So there is a Lemma of consensus stability to system (3).

Lemma 2 Let α be an orthonormal matrix in $R^{n\times p}$, and define the orthonormal complement of the matrix α as $\alpha_\perp$, and $\alpha_\perp^{\rm T}\alpha = 0$. For any initial condition, the state $x(t)$ in system (3) achieves consensus if and only if 1) $-L\alpha = 0$, and 2) there exists X such that

$$\alpha_\perp^{\rm T}((-L)X + X\,(-L)^{\rm T})\alpha_\perp < 0. \tag{5}$$

The relevant proof may be found in Ref.[23].

3.2. Consensus protocol

Now we investigate the consensus equilibrium for the special case where the communication topology is fixed. With the consensus algorithm (1), the final consensus value is given by $x^* = \sum_{i=1}^{n} \gamma_i x_i(0)$, where $\gamma = [\,\gamma_1,\ \ldots,\ \gamma_n\,]^{\rm T}$ with $\gamma_i \geq 0$ being a nonnegative left eigenvector of $-L$ associated with eigenvalue 0, and $\sum_{i=1}^{n} \gamma_i = 1$. Note that $\gamma_i \neq 0$ if agent i has a directed path to all the other agents in the information exchange and $\gamma_i = 0$ if there does not exist such a directed path. Based on Refs.[9, 11], we have the following theorem.

Theorem 1 Assume that $G = (v, \varepsilon)$ is a strongly connected digraph with Laplacian $-L$ and let γ be a nonnegative column left eigenvector of $-L$ corresponding to the zero eigenvalue, then we will have $\lim_{t\to\infty} {\rm e}^{-Lt} \to \alpha\gamma^{\rm T}$ and $x_i\,(t) \to \sum_{j=1}^{n} \gamma_j x_j\,(0)$ as $t \to \infty$, where $\sum_{j=1}^{n} \gamma_j = 1$.

Proof Let J be the Jordan form associated with $-L$, i.e. $-L = SJS^{-1}$. Thus $\exp(-Lt) = S\ \exp(Jt)S^{-1}$. As $t \to \infty$, it can be verified that the $\exp(Jt)$ converges to a matrix $Q = [q_{ij}]$ with a single nonzero element $q_{11} = 1$

$$Q = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}.$$

The fact that the other blocks in the diagonal of $\exp(Jt)$ vanish is due to the property that ${\rm Re}\,(\lambda_k\,(-L)) < 0$ for all $k \geq 2$ where $\lambda_k(-L)$ is the $k^{\rm th}$-largest eigenvalue of $-L$ in terms of magnitude $|\lambda_k|$. With some manipulations in Ref.[24], it is obtained that

$$\lim_{t\to\infty} {\rm e}^{-Lt} \to \alpha\gamma^{\rm T},$$

where $\gamma_j, j = 1, \ldots, n$ correspond to the first row of matrix S^{-1}. The equation of $\sum_{j=1}^{n} \gamma_j = 1$ comes from the proposition that $\exp(-Lt)$ has a row sum equal to 1 for any t.

Corollary 1 If the communication topology is fixed and strongly connected, then the consensus equilibrium state is a convex combination of the initial information states.

Theorem 2 The convex combination coefficients of equilibrium state in system (3) satisfy the following linear equations:

$$\begin{cases} \gamma^{\rm T} L = 0, \\ \sum_{j=1}^{n} \gamma_j = 1, \end{cases} \tag{6}$$

where $\gamma = [\,\gamma_1,\ \ldots,\ \gamma_n\,]^{\mathrm{T}}$ is variable, and it is also the left eigenvector of L corresponding to the zero eigenvalue.

Proof The left eigenvector $\gamma = [\,\gamma_1,\ \ldots,\ \gamma_n\,]^{\mathrm{T}}$ and right eigenvector $\alpha = [1, \ldots, 1]^{\mathrm{T}}$ of the L corresponding to the zero eigenvalue satisfy $\gamma^{\mathrm{T}}\alpha = 1$, $\gamma^{\mathrm{T}}L = 0$ and $L\alpha = 0$. Because the $\mathrm{Rank}(L) = n - 1$, the homogeneous linear equation $\gamma^{\mathrm{T}}L = 0$ may have infinitely many solutions. But the extended equation of Eq.(6) is a non-homogeneous equation, and in which both the rank of its coefficient matrix and the rank of its augmented matrix equal the number n of its unknowns. Thus non-homogeneous equation of Eq.(6) admits a unique solution. This unique non-zero solution is the left eigenvector of L corresponding to the zero eigenvalue.

4. Example and simulation study

Figure 2 is a connected digraph of order $n = 3$ with a set of nodes $v = \{v_1,\ v_2,\ v_3\}$ and a set of edges $\varepsilon = \{e_{12},\ e_{23},\ e_{31},\ e_{21}\}$. $\sum = \{\sum_i : i = 1, 2, 3\}$ represents cooperative agents at the nodes. We can obtain the graph Laplacian matrix as follows:

$$L = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix}.$$

Suppose that the initial state is $[\,0.2\ 0.4\ 0.6\,]^{\mathrm{T}}$, then we can obtain the final consensus protocol of three nodes based on system (3), where the equilibrium state satisfies

$$\lim_{t\to\infty} x(t) = [0.5x_1(0) + 0.25x_2(0) + 0.25x_3(0)] = 0.35.$$

It is shown in Fig.3.

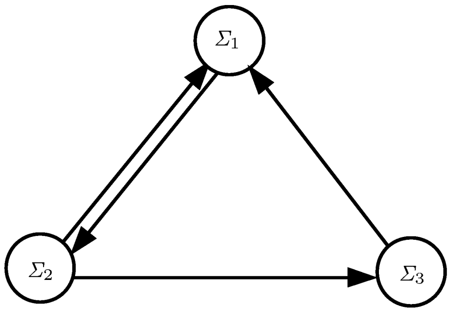

Fig.2. The communication topology of three agents.

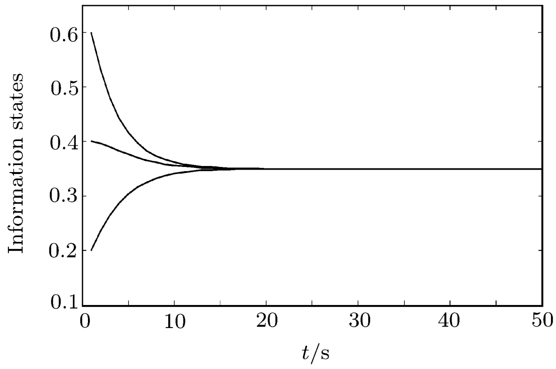

Fig.3. Consensus for three agents.

Shown in Fig.4 is a six-agent system. The initial state is $[\,0.2\ 0.7\ 0.3\ 0.6\ 0.4\ 0.5\,]^{\mathrm{T}}$, thus the corresponding graph Laplacian could be obtained as

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 2 & -1 \\ -1 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

The simulation result in Fig.5 shows that the consensus is obtained, and it can be verified that

$$\lim_{t\to\infty} x(t) = \left[\frac{7}{28}x_1(0) + \frac{3}{28}x_2(0) + \frac{3}{28}x_3(0) + \frac{6}{28}x_4(0) + \frac{5}{28}x_5(0) + \frac{4}{28}x_6(0)\right] = 0.4286.$$

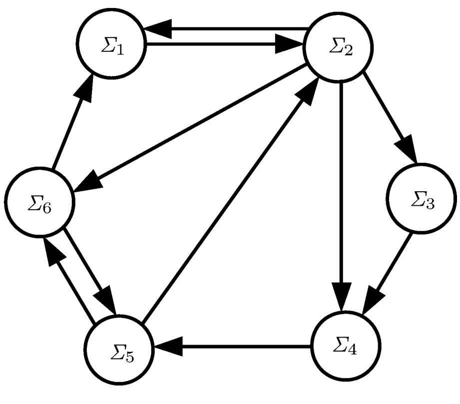

Fig.4. The communication topology of six agents.

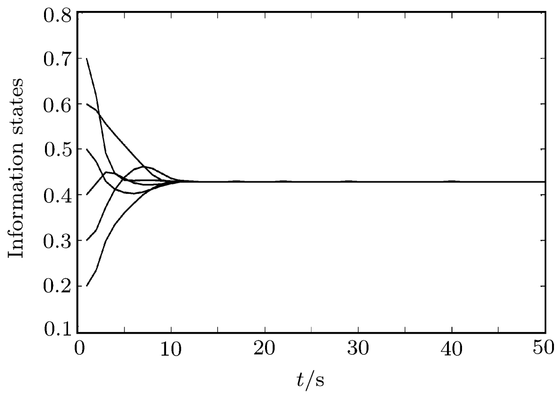

Fig.5. Consensus for six agents.

5. Conclusion

In this paper, a survey on consensus protocol for the time invariant multi-agent continuous system has been given. Based on the consensus protocol definition, we have analysed the stabilization of multi-agent systems and also studied the consensus algorithms of the networked systems with fixed topology. Furthermore, we have investigated three and six nodes of agent systems. The results of the calculating and simulation verify that the theory is correct.

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