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New robust stability results for bidirectional associative memory neural networks with multiple time delays

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ABSTRACT

In this paper, the robust stability problem is investigated for a class of bidirectional associative memory (BAM) neural networks with multiple time delays. By employing suitable Lyapunov functionals and using the upper bound norm for the interconnection matrices of the neural network system, some novel sufficient conditions ensuring the existence, uniqueness and global robust stability of the equilibrium point are derived. The obtained results impose constraint conditions on the system parameters of neural network independent of the delay parameters. Some numerical examples and simulation results are given to demonstrate the applicability and effectiveness of our results, and to compare the results with previous robust stability results derived in the literature.

1. Introduction

In recent years, neural networks have received considerable attention because of their successful applications in image processing, associative memories, optimization problems and other engineering areas [1,2]. Such applications rely on the qualitative stability properties of the designed neural network. Therefore, stability analysis of neural networks plays an important role in the designs and applications of neural networks. On the other hand, time delays occur in VLSI implementation of neural networks due to the finite switching speed of neuron amplifiers, and the finite speed of signal propagation. It is also known that the working with the delayed version of neural networks is important for solving some classes of motion-related optimization problems. However, it has been revealed that time delays may cause instability and oscillation of neural networks. For these reasons, it is of great importance to study the equilibrium and stability properties of neural networks in the presence of time delays. Some results concerning the dynamical behavior of various neural networks with or without delay has been reported in [3–20] and the references therein. We should also point out that, in hardware implementation of neural networks, the network parameters of the system may subject to some changes due to the tolerances of electronic components employed in the design. In such cases, it is desired that the stability properties of neural network should not be affected by the small deviations in the values of the parameters. In other words, the neural network must be globally robustly stable. Global robust stability of standard neural network models with time delays has been studied by many researchers and some important robust stability results have been reported in [21–26].

Bidirectional associative memory (BAM) neural networks were first introduced by Kosko [27,28]. A BAM neural network is composed of neurons arranged in two layers. The neurons in one layer are fully interconnected to the neurons in the other

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layer, while there are no interconnection among neurons in the same layer. It uses the forward and backward information flow to produce an associative search for stored stimulus–response association. One beneficial characteristic of the BAM is its ability to recall stored pattern pairs in the presence of noise. One may refer to [29] for detailed memory architecture and examples of BAM neural networks. This class of networks has successful application perspective in the field of pattern recognition and artificial intelligence due to its generalization of the single-layer auto-associative Hebbian correlator to a two-layer pattern-matched heteroassociative circuit [30]. Some of these applications require that there should be a well-defined computable solution for all possible initial states. From a mathematical point of view, this means that the equilibrium point of the designed neural network is globally asymptotically stable (GAS). The stability of the BAM neural networks has been extensively studied in the literature in the recent years and many different sufficient conditions ensuring the stability of BAM neural networks have been given in [31–43]. However, many of the existing stability results derived for the BAM neural networks can be applicable when only a pure delayed neural network model is employed. In recently published papers [44–48], a hybrid BAM neural network model in which both instantaneous and delayed signaling occur was considered.

In this paper, we study the equilibrium and robust stability properties of hybrid bidirectional associative memory neural networks with multiple time delays. By employing more general types of suitable Lyapunov–Krasovskii functionals and using the upper bound norm for the interconnection matrices of the neural system we obtain some novel delay-independent sufficient conditions for the existence, uniqueness and global robust asymptotic stability of the equilibrium point for hybrid, BAM neural networks with time delays. Some numerical examples are also given to prove that our conditions can be considered as the alternative results to the previous stability results derived in the literature.

2. Model description

Dynamical behavior of a hybrid BAM neural network with constant time delays is described by the following set of differential equations [47]:

$$\begin{aligned} \dot{u}_i(t) &= -a_i u_i(t) + \sum_{j=1}^{m} w_{ji} g_j(z_j(t)) + \sum_{j=1}^{m} w_{ji}^{\tau} g_j(z_j(t-\tau_{ji})) + I_i, \quad \forall i, \\ \dot{z}_j(t) &= -b_j z_j(t) + \sum_{i=1}^{n} v_{ij} g_i(u_i(t)) + \sum_{i=1}^{n} v_{ij}^{\tau} g_i(u_i(t-\sigma_{ij})) + J_j, \quad \forall j, \end{aligned} \tag{1}$$

The BAM neural network model (1) can be regarded as a neural network model having two layers. n denotes number of the neurons in the first layer and m denotes the number of neurons in the second layer. $u_i(t)$ is the state of the ith neuron in the first layer and $z_j(t)$ is the state of the jth neuron in the second layer. a_i and b_j denote the neuron charging time constants and passive decay rates, respectively; $w_{ji},\ w_{ji}^{\tau},\ v_{ij}$ and v_{ij}^{τ} are synaptic connection strengths; g_i and g_j represent the activation functions of the neurons and the propagational signal functions, respectively; and I_i and J_j are the exogenous inputs.

It will be assumed that $a_i, b_j,\ w_{ji},\ w_{ji}^{\tau},\ v_{ij},\ v_{ij}^{\tau},\ \tau_{ji}$ and σ_{ij} in system (1) are uncertain but bounded, and belong to the following intervals:

$$\begin{aligned} A_I &:= \{A = diag(a_i) : 0 < \underline{A} \leqslant A \leqslant \overline{A}, i.e., 0 < \underline{a}_i \leqslant a_i \leqslant \overline{a}_i,\ i = 1,2,\ldots,n,\ \forall A \in A_I\}, \\ B_I &:= \{B = diag(b_j) : 0 < \underline{B} \leqslant B \leqslant \overline{B}, i.e., 0 < \underline{b}_j \leqslant b_j \leqslant \overline{b}_j,\ j = 1,2,\ldots,m,\ \forall B \in B_I\}, \\ W_I &:= \{W = (w_{ji})_{m\times n} : \underline{W} \leqslant W \leqslant \overline{W}, i.e., \underline{w}_{ji} \leqslant w_{ji} \leqslant \overline{w}_{ji},\ i = 1,2,\ldots,n;\ j = 1,2,\ldots,m,\ \forall W \in W_I\}, \\ V_I &:= \{V = (v_{ij})_{n\times m} : \underline{V} \leqslant V \leqslant \overline{V}, i.e., \underline{v}_{ij} \leqslant v_{ij} \leqslant \overline{v}_{ij},\ i = 1,2,\ldots,n;\ j = 1,2,\ldots,m,\ \forall V \in V_I\}, \\ W_I^{\tau} &:= \{W^{\tau} = (w_{ji}^{\tau})_{m\times n} : \underline{W^{\tau}} \leqslant W \leqslant \overline{W}^{\tau}, i.e., \underline{w}_{ji}^{\tau} \leqslant w_{ji}^{\tau} \leqslant \overline{w}_{ji}^{\tau},\ i = 1,2,\ldots,n;\ j = 1,2,\ldots,m,\ \forall W^{\tau} \in W_I^{\tau}\}, \\ V_I^{\tau} &:= \{V^{\tau} = (v_{ij}^{\tau})_{n\times m} : \underline{V}^{\tau} \leqslant V \leqslant \overline{V}^{\tau}, i.e., \underline{v}_{ij}^{\tau} \leqslant v_{ij}^{\tau} \leqslant \overline{v}_{ij}^{\tau},\ i = 1,2,\ldots,n;\ j = 1,2,\ldots,m,\ \forall V^{\tau} \in V_I^{\tau}\}, \\ \tau_I &:= \{\tau = (\tau_{ji})_{m\times n} : \underline{\tau} \leqslant \tau \leqslant \overline{\tau}, i.e., \underline{\tau}_{ji} \leqslant \tau_{ji} \leqslant \overline{\tau}_{ji},\ i = 1,2,\ldots,n;\ j = 1,2,\ldots,m,\ \forall \tau \in \tau_I\}, \\ \sigma_I &:= \{\sigma = (\sigma_{ij})_{n\times m} : \underline{\sigma} \leqslant \sigma \leqslant \overline{\sigma}, i.e., \underline{\sigma}_{ij} \leqslant \sigma_{ij} \leqslant \overline{\sigma}_{ij},\ i = 1,2,\ldots,n;\ j = 1,2,\ldots,m,\ \forall \sigma \in \sigma_I\}. \end{aligned} \tag{2}$$

In order to establish the desired stability properties of neural network model (1), it is first necessary to specify the class of activation functions. The activation functions we employ in (1) are assumed to satisfy the following conditions:

(H1) There exist some positive constants $\ell_i,\ i = 1,2,\ldots,n$ and $k_j,\ j = 1,2,\ldots,m$ such that

$$0 \leqslant \frac{g_i(\bar{x}) - g_i(\bar{y})}{\bar{x} - \bar{y}} \leqslant \ell_i, \quad 0 \leqslant \frac{g_j(\hat{x}) - g_j(\hat{y})}{\hat{x} - \hat{y}} \leqslant k_j$$

for all $\bar{x}, \bar{y}, \hat{x}, \hat{y} \in R$. This class of functions will be denoted by $g \in \mathcal{K}$.

(H2) There exist positive constants $M_i,\ i = 1,2,\ldots,n$ and $L_j,\ j = 1,2,\ldots,m$ such that $|g_i(u)| \leqslant M_i$ and $|g_j(z)| \leqslant L_j$ for all $u, z \in R$. Note that this assumption implies that the activation functions are bounded and this class of functions will be denoted by $g \in \mathcal{B}$.

3. Preliminaries

Let $v = (v_1, v_2, \ldots, v_n)^T \in R^n$ be a column vector and $Q = (q_{ij})_{nxn}$ be a real matrix. The three commonly used vector norms $\|v\|_1, \|v\|_2, \|v\|_\infty$are defined as:

$$\|v\|_1 = \sum_{i=1}^{n}|v_i|, \quad \|v\|_2 = \sqrt{\sum_{i=1}^{n} v_i^2}, \quad \|v\|_\infty = \max_{1\leqslant i\leqslant n}|v_i|.$$

The three commonly used matrix norms $\|Q\|_1, \|Q\|_2, \|Q\|_\infty$ are defined as follows:

$$\|Q\|_1 = \max_{1\leqslant j\leqslant n}\sum_{i=1}^{n}|q_{ij}|, \quad \|Q\|_2 = [\lambda_M(Q^TQ)]^{1/2},$$

$$\|Q\|_\infty = \max_{1\leqslant i\leqslant n}\sum_{j=1}^{n}|q_{ij}|.$$

For the vector $v = (v_1, v_2, \ldots, v_n)^T, |v|$ will denote $v = (|v_1|, |v_2|, \ldots, |v_n|)^T$. For the matrix $Q = (q_{ij})_{nxn}$, the matrix $|Q|$ will denote $|Q| = (|q_{ij}|)_{nxn}$, and $\lambda_m(Q)$ and $\lambda_M(Q)$ will denote the minimum and maximum eigenvalues of Q, respectively. If $P = (p_{ij})_{nxn}$ and $Q = (q_{ij})_{nxn}$ are two real symmetric matrices, then $P \leqslant Q$ will imply that $v^TPv \leqslant v^TQv$ for all $v = (v_1, v_2, \ldots, v_n)^T \in R^n$.

Lemma 1 [3]. *For $A \in A_I := \{A = (a_{ij}) : \underline{A} \leqslant A \leqslant \overline{A},\ i.e., \underline{a}_{ij} \leqslant a_{ij} \leqslant \overline{a}_{ij},\ i,j = 1,2,\ldots,n\}$, the following inequality holds:*

$$\|A\|_2^2 \leqslant \|A^*\|_2^2 + \|A_*\|_2^2 + 2\|A_*^T|A^*|\|_2,$$

where $A^* = \frac{1}{2}(\overline{A} + \underline{A}),\ A_* = \frac{1}{2}(\overline{A} - \underline{A})$.

Lemma 2 [21]. *For any matrix $A \in [\underline{A}, \overline{A}]$, the following inequality holds:*

$$||A||_2 \leqslant ||A^*||_2 + ||A_*||_2,$$

where $A^* = \frac{1}{2}(\overline{A} + \underline{A}),\ A_* = \frac{1}{2}(\overline{A} - \underline{A})$.

Lemma 3. *For any two vectors $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ and $\upsilon = (\upsilon_1, \upsilon_2, \ldots, \upsilon_n)^T$, the following equality holds:*

$$2\omega^T\upsilon = 2\upsilon^T\omega \leqslant \gamma\omega^T\omega + \frac{1}{\gamma}\upsilon^T\upsilon,$$

where γ is any positive constant.

4. Global robust stability results

In this section, we present some theorems proving the conditions that guarantee the global asymptotic stability of the equilibrium point of neural system (1). Under Assumption (H2), neural network defined by (1) always has an equilibrium point. Therefore, what we need to prove is the global asymptotic stability of the equilibrium point. To this end, the equilibrium point of system (1) will be shifted to the origin. By using the transformation

$$x_i(\cdot) = u_i(\cdot) - u_i^*, \quad i = 1, 2, \ldots, n,$$

$$y_j(\cdot) = z_j(\cdot) - z_j^*, \quad j = 1, 2, \ldots, m,$$

system (1) can be transformed into the following form:

$$\begin{aligned} \dot{x}_i(t) &= -a_ix_i(t) + \sum_{j=1}^{m} w_{ji}f_j(y_j(t)) + \sum_{j=1}^{m} w_{ji}^{\tau}f_j(y_j(t - \tau_{ji})), \quad \forall i, \\ \dot{y}_j(t) &= -b_jy_j(t) + \sum_{i=1}^{n} v_{ij}f_i(x_i(t)) + \sum_{i=1}^{n} v_{ij}^{\tau}f_i(x_i(t - \sigma_{ij})), \quad \forall j, \end{aligned} \tag{3}$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$, $y(t) = (y_1(t), y_2(t), \ldots, y_n(t))^T$, $f(x(t)) = (f_1(x_1(t)), f_2(x_2(t)), \ldots, f_n(x_n(t)))^T$, $f(y(t)) = (f_1(y_1(t)), f_2(y_2(t)), \ldots, f_n(y_n(t)))^T$, $f(x(t - \sigma)) = (f_1(x_1(t - \sigma_1)), f_2(x_2(t - \sigma_2)), \ldots, f_n(x_n(t - \sigma_n)))^T$, $f(y(t - \tau)) = (f_1(y_1(t - \tau_1)), f_2(y_2(t - \tau_2)), \ldots, f_n(y_n(t - \tau_n)))^T$. The functions $f_i(x_i), f_j(y_j)$ are of the form:

$$f_i(x_i(\cdot)) = g_i(x_i(\cdot) + u_i^*) - g_i(u_i^*), \quad i = 1, 2, \ldots, n,$$

$$f_j(y_j(\cdot)) = g_j(y_j(\cdot) + z_j^*) - g_j(z_j^*), \quad j = 1, 2, \ldots, m.$$

It can be verified that the functions f_i and f_j satisfy the assumptions on g_i and g_j, i.e., $g_i \in \mathcal{K}$ and $g_j \in \mathcal{B}$ implies that $f_i \in \mathcal{K}$ and $f_j \in \mathcal{B}$, respectively. We also note that $f_i(0) = 0$ and $f_j(0) = 0,\ i = 1, 2, \ldots, n$.

Note that the equilibrium point of system (1) is globally asymptotically stable, if the origin of system (4) is a globally asymptotically stable equilibrium point. Therefore, in order to prove the global asymptotic stability of the equilibrium point of system (1), it will be sufficient to prove the global asymptotic stability of the origin of system (4). We can now proceed with the following result:

Theorem 1. *Let the activation functions satisfy assumptions (H1) and (H2). Then, neural system (1) with (2) has a unique equilibrium point which is globally asymptotically robustly stable if there exist positive constants α and β such that the network parameters of the system satisfy the following conditions*

$$\delta_i = m(2\underline{a}_i - \alpha - \beta) - \frac{1}{\beta} n\ell_i^2 (\|V^*\|_2^2 + \|V_*\|_2^2 + 2\|V_*^T|V^*|\|_2) - \frac{1}{\alpha} n^2 \ell_i^2 \sum_{j=1}^{m} (v_{ij}^{\tau^*})^2 > 0, \quad \forall i > 0,$$

$$\Omega_j = n(2\underline{b}_j - \alpha - \beta) - \frac{1}{\beta} mk_j^2 (\|W^*\|_2^2 + \|W_*\|_2^2 + 2\|W_*^T|W^*|\|_2) - \frac{1}{\alpha} m^2 k_j^2 \sum_{i=1}^{n} (w_{ji}^{\tau^*})^2 > 0, \quad \forall j > 0,$$

where $W = (w_{ji}),\ V = (v_{ij}),\ W^* = \frac{1}{2}(\overline{W} + \underline{W}),\ W_* = \frac{1}{2}(\overline{W} - \underline{W}),\ V^* = \frac{1}{2}(\overline{V} + \underline{V}),\ V_* = \frac{1}{2}(\overline{V} - \underline{V}),\ v_{ij}^{\tau^*} = max\{|\underline{v}_{ij}^{\tau}|, |\overline{v}_{ij}^{\tau}|\}$ *and* $w_{ji}^{\tau^*} = max\{|\underline{w}_{ji}^{\tau}|, |\overline{w}_{ji}^{\tau}|\}$.

Proof. Define the following positive definite Lyapunov functional:

$$V(x(t), y(t)) = \sum_{i=1}^{n} mx_i^2(t) + \sum_{j=1}^{m} ny_j^2(t) + \frac{1}{\alpha} \sum_{i=1}^{n} \sum_{j=1}^{m} m^2 \left(w_{ji}^{\tau}\right)^2 \int_{t-\tau_{ji}}^{t} s_j^2(y_j(\eta)) d\eta + \frac{1}{\alpha} \sum_{j=1}^{m} \sum_{i=1}^{n} n^2 \left(v_{ij}^{\tau}\right)^2 \int_{t-\sigma_{ij}}^{t} s_i^2(x_i(\xi)) d\xi.$$

The derivative of $V(x(t), y(t))$ along the trajectories of the system is obtained as:

$$\begin{aligned}
\dot{V}(x(t), y(t)) = & -\sum_{i=1}^{n} 2ma_i x_i^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{m} 2mx_i(t) w_{ji} s_j(y_j(t)) + \sum_{i=1}^{n}\sum_{j=1}^{m} 2mx_i(t) w_{ji}^{\tau} s_j(y_j(t-\tau_{ji})) - \sum_{j=1}^{m} 2nb_j y_j^2(t) \\
& + \sum_{j=1}^{m}\sum_{i=1}^{n} 2ny_j(t) v_{ij} s_i(x_i(t)) + \sum_{j=1}^{m}\sum_{i=1}^{n} 2ny_j(t) v_{ij}^{\tau} s_i(x_i(t-\sigma_{ij})) + \frac{1}{\alpha}\sum_{i=1}^{n}\sum_{j=1}^{m} m^2 \left(w_{ji}^{\tau}\right)^2 s_j^2(y_j(t)) \\
& - \frac{1}{\alpha}\sum_{i=1}^{n}\sum_{j=1}^{m} m^2 \left(w_{ji}^{\tau}\right)^2 s_j^2(y_j(t-\tau_{ji})) + \frac{1}{\alpha}\sum_{j=1}^{m}\sum_{i=1}^{n} n^2 \left(v_{ij}^{\tau}\right)^2 s_i^2(x_i(t)) - \frac{1}{\alpha}\sum_{j=1}^{m}\sum_{i=1}^{n} n^2 \left(v_{ij}^{\tau}\right)^2 s_i^2(x_i(t-\sigma_{ij})) \\
\leqslant & -\sum_{i=1}^{n} 2ma_i x_i^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{m} 2mx_i(t) w_{ji} s_j(y_j(t)) + \sum_{i=1}^{n}\sum_{j=1}^{m} 2mx_i(t) w_{ji}^{\tau} s_j(y_j(t-\tau_{ji})) - \sum_{j=1}^{m} 2nb_j y_j^2(t) \\
& + \sum_{j=1}^{m}\sum_{i=1}^{n} 2ny_j(t) v_{ij} s_i(x_i(t)) + \sum_{j=1}^{m}\sum_{i=1}^{n} 2ny_j(t) v_{ij}^{\tau} s_i(x_i(t-\sigma_{ij})) + \frac{1}{\alpha}\sum_{i=1}^{n}\sum_{j=1}^{m} m^2 (w_{ji}^{\tau})^2 k_j^2 y_j^2(t) \\
& - \frac{1}{\alpha}\sum_{i=1}^{n}\sum_{j=1}^{m} m^2 (w_{ji}^{\tau})^2 s_j^2(y_j(t-\tau_{ji})) + \frac{1}{\alpha}\sum_{j=1}^{m}\sum_{i=1}^{n} n^2 (v_{ij}^{\tau})^2 \ell_i^2 x_i^2(t) - \frac{1}{\alpha}\sum_{j=1}^{m}\sum_{i=1}^{n} n^2 (v_{ij}^{\tau})^2 s_i^2(x_i(t-\sigma_{ij}))
\end{aligned} \tag{4}$$

We note the following inequalities:

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{m} 2mx_i(t) w_{ji} s_j(y_j(t)) = 2mx^T(t) WS(y(t)) & \leqslant m\beta x^T(t)x(t) + m\frac{1}{\beta} S^T(y(t)) W^T WS(y(t)) \\
& \leqslant m\beta x^T(t)x(t) + m\frac{1}{\beta} ||W||_2^2 ||S(y(t))||_2^2 \leqslant m\beta \sum_{i=1}^{n} x_i^2(t) + m\frac{1}{\beta} ||W||_2^2 \sum_{j=1}^{m} k_j^2 y_j^2(t),
\end{aligned} \tag{5}$$

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{i=1}^{n} 2ny_j(t) v_{ij} s_i(x_i(t)) = 2ny^T(t) VS(x(t)) & \leqslant n\beta y^T(t)y(t) + n\frac{1}{\beta} S^T(x(t)) V^T VS(x(t)) \\
& \leqslant n\beta y^T(t)y(t) + n\frac{1}{\beta} ||V||_2^2 ||S(x(t))||_2^2 \leqslant n\beta \sum_{j=1}^{m} y_j^2(t) + n\frac{1}{\beta} ||V||_2^2 \sum_{i=1}^{n} \ell_i^2 x_i^2(t),
\end{aligned} \tag{6}$$

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{m} 2mx_i(t) w_{ji}^{\tau} s_j(y_j(t-\tau_{ji})) & \leqslant \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha x_i^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{m} \frac{1}{\alpha} m^2 \left(w_{ji}^{\tau}\right)^2 s_j^2(y_j(t-\tau_{ji})) \\
& = m\alpha \sum_{i=1}^{n} x_i^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{m} \frac{1}{\alpha} m^2 \left(w_{ji}^{\tau}\right)^2 s_j^2(y_j(t-\tau_{ji})),
\end{aligned} \tag{7}$$

$$\sum_{j=1}^{m}\sum_{i=1}^{n}2ny_j(t)v_{ij}^{\tau}s_i(x_i(t-\sigma_{ij})) \leqslant \sum_{j=1}^{m}\sum_{i=1}^{n}\alpha y_j^2(t)+\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{1}{\alpha}n^2\left(v_{ij}^{\tau}\right)^2 s_i^2(x_i(t-\sigma_{ij}))$$
$$= n\alpha\sum_{j=1}^{m}y_j^2(t)+\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{1}{\alpha}n^2\left(v_{ij}^{\tau}\right)^2 s_i^2(x_i(t-\sigma_{ij})). \quad (8)$$

Using (5)–(8) in (4) results in

$$\dot V(x(t),y(t)) \leqslant -\sum_{i=1}^{n}2ma_ix_i^2(t)+m\beta\sum_{i=1}^{n}x_i^2(t)+m\frac{1}{\beta}||W||_2^2\sum_{j=1}^{m}k_j^2y_j^2(t)-\sum_{j=1}^{m}2nb_jy_j^2(t)+n\beta\sum_{j=1}^{m}y_j^2(t)+n\frac{1}{\beta}||V||_2^2\sum_{i=1}^{n}\ell_i^2x_i^2(t)$$
$$+m\alpha\sum_{i=1}^{n}x_i^2(t)+n\alpha\sum_{j=1}^{m}y_j^2(t)+\frac{1}{\alpha}\sum_{i=1}^{n}\sum_{j=1}^{m}m^2\left(w_{ji}^{\tau}\right)^2k_j^2y_j^2(t)+\frac{1}{\alpha}\sum_{j=1}^{m}\sum_{i=1}^{n}n^2\left(v_{ij}^{\tau}\right)^2\ell_i^2x_i^2(t).$$

Since $\|W\|_2^2 \leqslant \|W^*\|_2^2+\|W_*\|_2^2+2\|W_*^T|W^*|\|_2,\ \|V\|_2^2 \leqslant \|V^*\|_2^2+\|V_*\|_2^2+2\|V_*^T|V^*|\|_2$ and $(w_{ji}^{\tau})^2 \leqslant (w_{ji}^{\tau^*})^2,\ \left(v_{ji}^{\tau}\right)^2 \leqslant \left(v_{ji}^{\tau^*}\right)^2$

$$\dot V(x(t),y(t)) \leqslant \sum_{i=1}^{n}\left\{m(-2\underline{a}_i+\alpha+\beta)+\frac{1}{\beta}n\ell_i^2\left(\|V^*\|_2^2+\|V_*\|_2^2+2\|V_*^T|V^*|\|_2\right)+\frac{1}{\alpha}n^2\ell_i^2\sum_{j=1}^{m}\left(v_{ij}^{\tau^*}\right)^2\right\}x_i^2(t)$$
$$+\sum_{j=1}^{m}\left\{n(-2\underline{b}_j+\alpha+\beta)+\frac{1}{\beta}mk_j^2\left(\|W^*\|_2^2+\|W_*\|_2^2+2\|W_*^T|W^*|\|_2\right)+\frac{1}{\alpha}m^2k_j^2\sum_{i=1}^{n}\left(w_{ji}^{\tau^*}\right)^2\right\}y_j^2(t)$$
$$= -\sum_{i=1}^{n}\delta_ix_i^2(t)-\sum_{j=1}^{m}\Omega_jy_j^2(t).$$

Since $\delta_i > 0$ for $i = 1,\dots,n$ and $\Omega_j > 0$ for $j = 1,\dots,m$, it follows that $\dot V(x(t),y(t)) < 0$ for $x(t) \neq 0$ or $y(t) \neq 0$. Hence, by the standard Lyapunov-type theorem in functional differential equations we can conclude that the origin of system (3) is globally asymptotically stable.

Theorem 2. *Let the activation functions satisfy assumptions* (H1) *and* (H2). *Then, neural system* (1) *with* (2) *has a unique equilibrium point which is globally asymptotically robustly stable if there exist positive constants* α *and* β *such that the network parameters of the system satisfy the following conditions*

$$\varphi_i = m(2\underline{a}_i-\alpha\ell_i^2-\beta)-\frac{1}{\beta}n\ell_i^2(\|V^*\|_2^2+\|V_*\|_2^2+2\|V_*^T|V^*|\|_2)-\frac{1}{\alpha}m^2\sum_{j=1}^{m}(w_{ji}^{\tau^*})^2 > 0, \quad \forall i,$$

$$\vartheta_j = n(2\underline{b}_j-\alpha k_j^2-\beta)-\frac{1}{\beta}mk_j^2(\|W^*\|_2^2+\|W_*\|_2^2+2\|W_*^T|W^*|\|_2)-\frac{1}{\alpha}n^2\sum_{i=1}^{n}(v_{ij}^{\tau^*})^2 > 0, \quad \forall j,$$

where $W = (w_{ji}),\ V = (v_{ij}), W^* = \frac{1}{2}(\overline{W}+\underline{W}),\ W_* = \frac{1}{2}(\overline{W}-\underline{W}),\ V^* = \frac{1}{2}(\overline{V}+\underline{V}),\ V_* = \frac{1}{2}(\overline{V}-\underline{V}),\ v_{ij}^{\tau^*} = max\{|\underline{v}_{ij}^{\tau}|,|\overline{v}_{ij}^{\tau}|\}$ *and* $w_{ji}^{\tau^*} = max\{|\underline{w}_{ji}^{\tau}|,|\overline{w}_{ji}^{\tau}|\}$.

Proof. Define the following positive definite Lyapunov functional:

$$V(x(t),y(t)) = \sum_{i=1}^{n}mx_i^2(t)+\sum_{j=1}^{m}ny_j^2(t)+\alpha\sum_{i=1}^{n}\sum_{j=1}^{m}\int_{t-\tau_{ji}}^{t}s_j^2(y_j(\eta))d\eta+\alpha\sum_{j=1}^{m}\sum_{i=1}^{n}\int_{t-\sigma_{ij}}^{t}s_i^2(x_i(\xi))d\xi.$$

The derivative of $V(x(t),y(t))$ along the trajectories of the system is obtained as:

$$\dot V(x(t),y(t)) = -\sum_{i=1}^{n}2ma_ix_i^2(t)+\sum_{i=1}^{n}\sum_{j=1}^{m}2mx_i(t)w_{ji}s_j(y_j(t))+\sum_{i=1}^{n}\sum_{j=1}^{m}2mx_i(t)w_{ji}^{\tau}s_j(y_j(t-\tau_{ji}))-\sum_{j=1}^{m}2nb_jy_j^2(t)$$
$$+\sum_{j=1}^{m}\sum_{i=1}^{n}2ny_j(t)v_{ij}s_i(x_i(t))+\sum_{j=1}^{m}\sum_{i=1}^{n}2ny_j(t)v_{ij}^{\tau}s_i(x_i(t-\sigma_{ij}))+\alpha\sum_{i=1}^{n}\sum_{j=1}^{m}s_j^2(y_j(t))-\alpha\sum_{i=1}^{n}\sum_{j=1}^{m}s_j^2(y_j(t-\tau_{ji}))$$
$$+\alpha\sum_{j=1}^{m}\sum_{i=1}^{n}s_i^2(x_i(t))-\alpha\sum_{j=1}^{m}\sum_{i=1}^{n}s_i^2(x_i(t-\sigma_{ij})). \quad (9)$$

We also note that

$$\sum_{i=1}^{n}\sum_{j=1}^{m}2mx_i(t)w_{ji}^{\tau}s_j(y_j(t-\tau_{ji})) \leqslant \sum_{i=1}^{n}\sum_{j=1}^{m}\frac{1}{\alpha}m^2(w_{ji}^{\tau})^2x_i^2(t)+\sum_{i=1}^{n}\sum_{j=1}^{m}\alpha s_j^2(y_j(t-\tau_{ji})), \quad (10)$$

$$\sum_{j=1}^{m}\sum_{i=1}^{n}2ny_j(t)v_{ij}^{\tau}s_i(x_i(t-\sigma_{ij}))\leqslant\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{1}{\alpha}n^2(v_{ij}^{\tau})^2y_j^2(t)+\sum_{j=1}^{m}\sum_{i=1}^{n}\alpha s_i^2(x_i(t-\sigma_{ij})).\tag{11}$$

Using (5), (6), (10) and (11) in (9) leads to:

$$\dot V(x(t),y(t))\leqslant-\sum_{i=1}^{n}2ma_ix_i^2(t)+m\beta\sum_{i=1}^{n}x_i^2(t)+m\frac{1}{\beta}||W||_2^2\sum_{j=1}^{m}k_j^2y_j^2(t)-\sum_{j=1}^{m}2nb_jy_j^2(t)+n\beta\sum_{j=1}^{m}y_j^2(t)+n\frac{1}{\beta}||V||_2^2\sum_{i=1}^{n}\ell_i^2x_i^2(t)$$
$$+\alpha n\sum_{j=1}^{m}k_j^2y_j^2(t)+\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{1}{\alpha}m^2(w_{ji}^{\tau})^2x_i^2(t)+\alpha m\sum_{i=1}^{n}\ell_i^2x_i^2(t)+\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{1}{\alpha}n^2\left(v_{ij}^{\tau}\right)^2y_j^2(t)$$

Since $\|W\|_2^2\leqslant\|W^*\|_2^2+\|W_*\|_2^2+2\|W_*^T|W^*|\|_2,\ \|V\|_2^2\leqslant\|V^*\|_2^2+\|V_*\|_2^2+2\|V_*^T|V^*|\|_2$ and $\left(w_{ji}^{\tau}\right)^2\leqslant\left(w_{ji}^{\tau^*}\right)^2,\ \left(v_{ji}^{\tau}\right)^2\leqslant\left(v_{ji}^{\tau^*}\right)^2$

$$\dot V(x(t),y(t))\leqslant\sum_{i=1}^{n}\left\{m(-2\underline{a}_i+\alpha\ell_i^2+\beta)+\frac{1}{\beta}n\ell_i^2(\|V^*\|_2^2+\|V_*\|_2^2+2\|V_*^T|V^*|\|_2)+\frac{1}{\alpha}m^2\sum_{j=1}^{m}\left(w_{ji}^{\tau^*}\right)^2\right\}x_i^2(t)$$
$$+\sum_{j=1}^{m}\left\{n(-2\underline{b}_j+\alpha k_j^2+\beta)+\frac{1}{\beta}mk_j^2\left(\|W^*\|_2^2+\|W_*\|_2^2+2\|W_*^T|W^*|\|_2\right)+\frac{1}{\alpha}n^2\sum_{i=1}^{n}\left(v_{ij}^{\tau^*}\right)^2\right\}y_j^2(t)$$
$$=-\sum_{i=1}^{n}\varphi_ix_i^2(t)-\sum_{j=1}^{m}\vartheta_jy_j^2(t),$$

in which $\dot V(x(t),y(t))<0$ for all $x(t)\neq0$ or $y(t)\neq0$. Hence, the origin of system (3) is globally asymptotically stable.

5. Comparisons and examples

In this section, the results obtained in this paper will be compared with the previous global robust stability results of BAM neural networks derived in the literature. In order to make the comparison precise, first the previous results will be restated:

Theorem 3 [47]. *Let the activation functions satisfy assumptions* $(H1)$ *and* $(H2)$*. Then, neural system (1) with (2) has a unique equilibrium point which is globally asymptotically robust stable if there exist positive constants* α *and* β *such that the network parameters of the system satisfy the following conditions*

$$\zeta_i=m(2\underline{a}_i-\alpha-\beta)-\frac{1}{\beta}n\ell_i^2(||V^*||_2+||V_*||_2)^2-\frac{1}{\alpha}n^2\ell_i^2\sum_{j=1}^{m}\left(v_{ij}^{\tau^*}\right)^2>0,\quad i=1,2,\ldots,n,$$

$$\xi_j=n(2\underline{b}_j-\alpha-\beta)-\frac{1}{\beta}mk_j^2(||W^*||_2+||W_*||_2)^2-\frac{1}{\alpha}m^2k_j^2\sum_{i=1}^{n}\left(w_{ji}^{\tau^*}\right)^2>0,\quad j=1,2,\ldots,m,$$

where $W=(w_{ji}),\ V=(v_{ij}),\ W^*=\frac{1}{2}(\overline{W}+\underline{W}),\ W_*=\frac{1}{2}(\overline{W}-\underline{W}),\ V^*=\frac{1}{2}(\overline{V}+\underline{V}),\ V_*=\frac{1}{2}(\overline{V}-\underline{V}),\ v_{ij}^{\tau^*}=max\{|\underline{v}_{ij}^{\tau}|,|\overline{v}_{ij}^{\tau}|\}$ *and* $w_{ji}^{\tau^*}=max\{|\underline{w}_{ji}^{\tau}|,|\overline{w}_{ji}^{\tau}|\}$.

Theorem 4 [47]. *Let the activation functions satisfy assumptions* $(H1)$ *and* $(H2)$*. Then, neural system (1) with (2) has a unique equilibrium point which is globally asymptotically robust stable if there exist positive constants* α *and* β *such that the network parameters of the system satisfy the following conditions*

$$\phi_i=m(2\underline{a}_i-\alpha\ell_i^2-\beta)-\frac{1}{\beta}n\ell_i^2(||V^*||_2+||V_*||_2)^2-\frac{1}{\alpha}m^2\sum_{j=1}^{m}(w_{ji}^{\tau^*})^2>0,\quad i=1,2,\ldots,n,$$

$$\psi_j=n(2\underline{b}_j-\alpha k_j^2-\beta)-\frac{1}{\beta}mk_j^2(||W^*||_2+||W_*||_2)^2-\frac{1}{\alpha}n^2\sum_{i=1}^{n}(v_{ij}^{\tau^*})^2>0,\quad j=1,2,\ldots,m,$$

where $W=(w_{ji}),\ V=(v_{ij}),\ W^*=\frac{1}{2}(\overline{W}+\underline{W}),\ W_*=\frac{1}{2}(\overline{W}-\underline{W}),\ V^*=\frac{1}{2}(\overline{V}+\underline{V}),\ V_*=\frac{1}{2}(\overline{V}-\underline{V}),\ v_{ij}^{\tau^*}=max\{|\underline{v}_{ij}^{\tau}|,|\overline{v}_{ij}^{\tau}|\}$ *and* $w_{ji}^{\tau^*}=max\{|\underline{w}_{ji}^{\tau}|,|\overline{w}_{ji}^{\tau}|\}$.

Theorem 5 [48]. *For the neural system defined by (1),* let the activation functions satisfy $(H1)$ *and* $(H2)$*, and network parameters satisfy (2).* Then, the origin of neural system *(1) is globally asymptotically stable if there exist positive constants* $p>0$ *and* $q>0$ *such that the following conditions hold:*

$$\theta_i = 2\underline{a}_i - p - \frac{1}{p}\ell_i^2(||V^*||_2 + ||V_*||_2)^2 - q\sum_{j=1}^{m} w_{ji}^{\tau^*} - \frac{1}{q}\ell_i^2\sum_{j=1}^{m} v_{ij}^{\tau^*} > 0, \quad i = 1,2,\ldots,n,$$

$$\gamma_j = 2\underline{b}_j - p - \frac{1}{p}k_j^2(||W^*||_2 + ||W_*||_2)^2 - q\sum_{i=1}^{n} v_{ij}^{\tau^*} - \frac{1}{q}k_j^2\sum_{i=1}^{n} w_{ji}^{\tau^*} > 0, \quad j = 1,2,\ldots,m,$$

where $W^* = \frac{1}{2}(\overline{W} + \underline{W}),\ W_* = \frac{1}{2}(\overline{W} - \underline{W}),\ V^* = \frac{1}{2}(\overline{V} + \underline{V}),\ V_* = \frac{1}{2}(\overline{V} - \underline{V}),\ v_{ij}^{\tau^*} = max\{|\underline{v}_{ij}^{\tau}|, |\overline{v}_{ij}^{\tau}|\}$ *and* $w_{ji}^{\tau^*} = max\{|\underline{w}_{ji}^{\tau}|, |\overline{w}_{ji}^{\tau}|\}$.

In order to show that the conditions we have obtained in Theorems 1 and 2 provide new different set of sufficient criteria for determining the equilibrium and stability properties of system (1) from those presented in [47,48], we consider the following example.

Example 1. Assume that the network parameters of neural system (1) are given as follows:

$$\underline{W} = \underline{V} = \begin{bmatrix} 0 & 2a & 2a & 2a \\ -2a & -2a & 2a & 2a \\ 2a & -2a & 2a & -2a \\ -2a & 2a & 2a & -2a \end{bmatrix}, \quad \overline{W} = \overline{V} = \begin{bmatrix} 0 & 2a & 2a & 2a \\ -2a & -2a & 2a & 2a \\ 2a & -2a & 2a & -2a \\ -2a & 2a & 2a & 0 \end{bmatrix},$$

$$\underline{W}^{\tau} = \underline{V}^{\tau} = \begin{bmatrix} -2a & -2a & -2a & -2a \\ -2a & -2a & -2a & -2a \\ -2a & -2a & -2a & -2a \\ -2a & -2a & -2a & -2a \end{bmatrix}, \quad \overline{W}^{\tau} = \overline{V}^{\tau} = \begin{bmatrix} 2a & 2a & 2a & 2a \\ 2a & 2a & 2a & 2a \\ 2a & 2a & 2a & 2a \\ 2a & 2a & 2a & 2a \end{bmatrix},$$

$$\underline{A} = A = \overline{A} = \underline{B} = B = \overline{B} = I,$$

$$\ell_1 = \ell_2 = \ell_3 = \ell_4 = k_1 = k_2 = k_3 = k_4 = 1,$$

where $a > 0$ is real number. The matrices $W^*, W_*, V^*, V_*, W_*^T|W^*|$ and $V_*^T|V^*|$ are obtained as follows:

$$W^* = V^* = \begin{bmatrix} 0 & 2a & 2a & 2a \\ -2a & -2a & 2a & 2a \\ 2a & -2a & 2a & -2a \\ -2a & 2a & 2a & -a \end{bmatrix}, \quad W_* = V_* = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \end{bmatrix},$$

$$W_*^T|W^*| = V_*^T|V^*| = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2a^2 & 2a^2 & 2a^2 & a^2 \end{bmatrix},$$

where $||W^*||_2 = ||V^*||_2 = 4,8399a$, $||W_*||_2 = ||V_*||_2 = a$ and $||W_*^T|W^*|||_2 = ||V_*^T|V^*|||_2 = 3,6056a^2$. Let $\alpha = \beta = \frac{1}{2}$. Then, we obtain

$$\delta_1 = \delta_2 = \delta_3 = \delta_4 = \Omega_1 = \Omega_2 = \Omega_3 = \Omega_4 = 4 - 8(31,6358a^2) - 512a^2 = 1 - (191,2716)a^2,$$

in which $a^2 < \frac{1}{191,2716}$ or equivalently $a < \frac{1}{13,8301}$ implies that $\delta_1 = \delta_2 = \delta_3 = \delta_4 = \Omega_1 = \Omega_2 = \Omega_3 = \Omega_4 > 0$. Hence, if $a < \frac{1}{13,8301}$ holds, then the conditions of Theorem 1 are satisfied. We will now check the conditions of Theorem 2 for the same network parameters. The conditions of Theorem 2 are obtained as follows:

$$\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4 = \vartheta_1 = \vartheta_2 = \vartheta_3 = \vartheta_4 = 4 - 8(31,6358a^2) - 512a^2 = 1 - (191,2716)a^2,$$

Obviously, $a < \frac{1}{13,8301}$ ensures that the conditions of Theorem 2 hold. We note here that, for the network parameters given in this example, Theorems 1 and 2 impose the same constraint conditions on the network parameters.

We will now check the results of Theorems 3 and 4 for this example. The conditions of Theorems 3 and 4 are obtained as follows:

$$\zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = \xi_1 = \xi_2 = \xi_3 = \xi_4 = \phi_1 = \phi_2 = \phi_3 = \phi_4 = \psi_1 = \psi_2 = \psi_3 = \psi_4 = 4 - 8(4,8399a + a)^2 - 512a^2$$
$$= 1 - (196,2088)a^2$$

from which the stability condition is obtained as $a^2 < \frac{1}{196,2088}$ or equivalently $a < \frac{1}{14,0075}$.

Remark 1. For this specific example, our results require that $a < \frac{1}{13,8301}$. On the other hand, the results of Theorems 3 and 4 hold if and only if $a < \frac{1}{14,0075}$. Therefore, for $\frac{1}{14,0075} \leqslant a < \frac{1}{13,8301}$, our conditions obtained in Theorems 1 and 2 are satisfied but the results of Theorems 3 and 4 do not hold. Hence, our results impose less conservative constraints on the network parameters of this example than the constraints imposed by the results given in [47].

For the same network parameters, we will now check the results of Theorem 5. Let $p = 5,8399a$ and $q = 2$. Then the conditions of Theorem 5 are obtained as follows:

$$\theta_1 = \theta_2 = \theta_3 = \theta_4 = \gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 2 - 31,6798a = 1 - 15,8399a$$

from which the stability condition is obtained as $a < \frac{1}{15,8399}$.

Remark 2. For this specific example, our results require that $a < \frac{1}{13,8301}$. On the other hand, the results of Theorems 5 hold if and only if $a < \frac{1}{15,8399}$. Therefore, for $\frac{1}{15,8399} \leqslant a < \frac{1}{13,8301}$, our conditions obtained in Theorems 1 and 2 are satisfied but the results of Theorems 5 do not hold. Hence, our results impose less conservative constraints on the network parameters of this example than the constraints imposed by the results given in [48].

In what follows, we give some simulation results for the sake of verification of our proposed results.

For the neural network parameters given in Example 1, we choose the following fixed network parameters that satisfy the condition $a < \frac{1}{13,8301}$:

$$W = V = W^{\tau} = V^{\tau} = \begin{bmatrix} 0 & 0.125 & 0.125 & 0.125 \\ -0.125 & -0.125 & 0.125 & 0.125 \\ 0.125 & -0.125 & 0.125 & -0.125 \\ -0.125 & 0.125 & 0.125 & 0 \end{bmatrix},$$

$$\tau_{11} = 0.5,\ \tau_{12} = 0.3,\ \tau_{13} = 0.2,\ \tau_{14} = 0.7,$$

$$\tau_{21} = 0.6,\ \tau_{22} = 0.4,\ \tau_{23} = 0.3,\ \tau_{24} = 0.1,$$

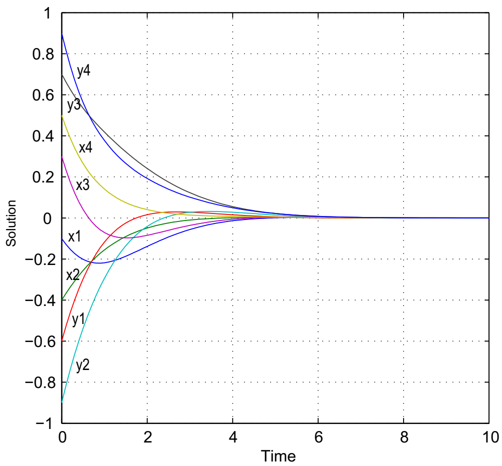
$$\tau_{31} = 0.8,\ \tau_{32} = 0.2,\ \tau_{33} = 0.9,\ \tau_{34} = 0.4,$$

$$\tau_{41} = 0.7,\ \tau_{42} = 0.1,\ \tau_{43} = 0.4,\ \tau_{44} = 0.5.$$

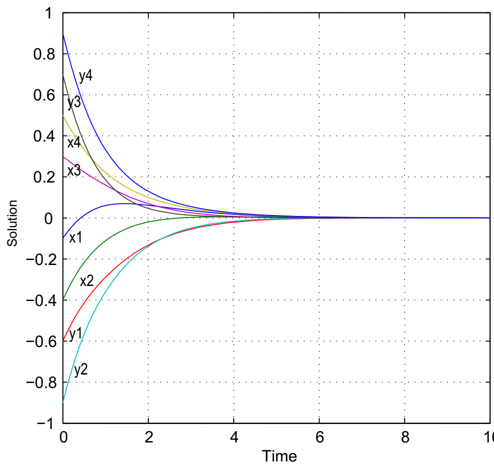
For this example, the Matlab simulation results are presented for different activation functions in Fig. 1(a) and Fig. 1(b).

For the same example, we now choose the following fixed network parameters that satisfying the constraint conditions imposed by our results:

$$W = V = W^{\tau} = V^{\tau} = \begin{bmatrix} 0 & 0.08 & 0.08 & 0.08 \\ -0.08 & -0.08 & 0.08 & 0.08 \\ 0.08 & -0.08 & 0.08 & -0.08 \\ -0.08 & 0.08 & 0.08 & -0.04 \end{bmatrix},$$

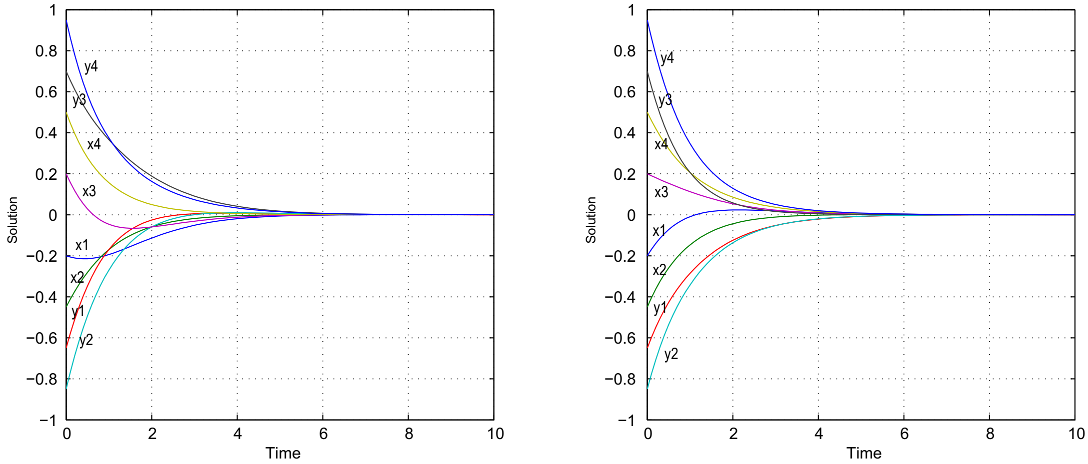


(a) Activation Functions $g(x(t)) = tanh(x(t))$, $g(y(t)) = tanh(y(t))$



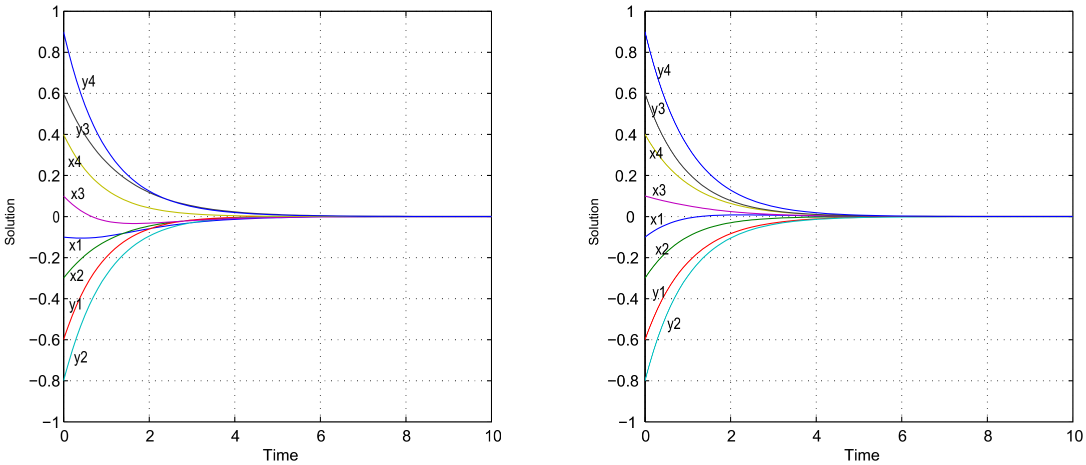
(b) Activation Functions $g(x(t)) = \frac{e^{-x(t)}-1}{e^{-x(t)}+1}$, $g(y(t)) = \frac{e^{-y(t)}-1}{e^{-y(t)}+1}$

Fig. 1. System solution for the initial states $x(0) = [-0.1\ \ -0.4\ 0.3\ 0.5]$ and $y(0) = [-0.6\ \ -0.9\ 0.7\ 0.9]$.



(a) Activation Functions $g(x(t)) = tanh(x(t))$, $g(y(t)) = tanh(y(t))$ (b) Activation Functions $g(x(t)) = \frac{e^{-x(t)}-1}{e^{-x(t)}+1}$, $g(y(t)) = \frac{e^{-y(t)}-1}{e^{-y(t)}+1}$

Fig. 2. System solution for the initial states $x(0) = [-0.2\ \ -0.45\ 0.2\ 0.5]$ and $y(0) = [-0.65\ \ -0.85\ 0.7\ 0.95]$.



(a) Activation Functions $g(x(t)) = tanh(x(t))$, $g(y(t)) = tanh(y(t))$ (b) Activation Functions $g(x(t)) = \frac{e^{-x(t)}-1}{e^{-x(t)}+1}$, $g(y(t)) = \frac{e^{-y(t)}-1}{e^{-y(t)}+1}$

Fig. 3. System solution for the initial states $x(0) = [-0.1\ \ -0.3\ 0.1\ 0.4]$ and $y(0) = [-0.6\ \ -0.8\ 0.6\ 0.9]$.

$$\tau_{11} = 0.5,\ \tau_{12} = 0.3,\ \tau_{13} = 0.2,\ \tau_{14} = 0.7,$$

$$\tau_{21} = 0.6,\ \tau_{22} = 0.4,\ \tau_{23} = 0.3,\ \tau_{24} = 0.1,$$

$$\tau_{31} = 0.8,\ \tau_{32} = 0.2,\ \tau_{33} = 0.9,\ \tau_{34} = 0.4,$$

$$\tau_{41} = 0.7,\ \tau_{42} = 0.1,\ \tau_{43} = 0.4,\ \tau_{44} = 0.5.$$

The Matlab simulation results for these parameters are given for different activation functions in Fig. 2(a) and Fig. 2(b).

For the same example, we now choose the following fixed network parameters that satisfying the constraint conditions imposed by our results:

$$W = V = W^{\tau} = V^{\tau} = \begin{bmatrix} 0 & 0.042 & 0.042 & 0.042 \\ -0.042 & -0.042 & 0.042 & 0.042 \\ 0.042 & -0.042 & 0.042 & -0.042 \\ -0.042 & 0.042 & 0.042 & -0.021 \end{bmatrix},$$

$$\tau_{11} = 0.2,\ \tau_{12} = 0.7,\ \tau_{13} = 0.4,\ \tau_{14} = 0.5,$$

$$\tau_{21} = 0.6,\ \tau_{22} = 0.1,\ \tau_{23} = 0.8,\ \tau_{24} = 0.9,$$

$$\tau_{31} = 0.3,\ \tau_{32} = 0.5,\ \tau_{33} = 0.7,\ \tau_{34} = 0.2,$$

$$\tau_{41} = 0.1,\ \tau_{42} = 0.8,\ \tau_{43} = 0.3,\ \tau_{44} = 0.6.$$

The Matlab simulation results for the above parameters are given for different activation functions in Fig. 3(a) and Fig. 3(b).

6. Conclusions

In this paper, by using the Lyapunov stability theory and the upper bound norm for the interconnection matrices of the neural system, novel sufficient conditions ensuring the existence, uniqueness and the global robust asymptotic stability of the equilibrium point have been derived for a class of hybrid bidirectional associative memory (BAM) neural networks with multiple time delays. The obtained stability results establish some relationships between the network parameters of neural network model independently of the delay parameters. A comparison between our results and the previous results implies that our results establish a new set of global robust asymptotic stability criteria for BAM neural networks with multiple time delays. In order to give some guidance for the future works in the area of robust stability of delayed neural networks, we need to point out that different and weaker upper bound norm estimation of interconnection matrices would be the key factor. Therefore, in order to improve the current robust stability results for neural networks, some more effort must be put into investigation of the interval matrix theory.

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