Robust Absolute Stability of Time-Varying Nonlinear Discrete-Time Systems

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***Abstract*—This paper studies the problem of robust absolute stability of a class of nonlinear discrete-time systems with time-varying matrix uncertainties of polyhedral type and multiple time-varying sector nonlinearities. By using the variational method and the Lyapunov second method, criteria for robust absolute stability are obtained in different forms for the class of systems under consideration. Specifically, we determine the parametric classes of Lyapunov functions which define the necessary and sufficient conditions of robust absolute stability. We apply the piecewise-linear Lyapunov functions of the infinity vector norm type to derive an algebraic criterion for robust absolute stability in the form of solvability conditions of a set of matrix equations. Some simple sufficient conditions of robust absolute stability are given which become necessary and sufficient for several special cases. An example is presented as an application of these results to a specific class of systems with time-varying interval matrices in the linear part.**

***Index Terms*—Absolute stability, difference inclusion, discrete-time systems, Lyapunov methods, robust stability, time-varying systems, variational method.**

I. Introduction

In the past two decades, considerable research efforts have been devoted to the study of robust control and robust stability of uncertain dynamic systems with parametric or nonparametric uncertainties. A great number of significant results covering these issues have been reported in the literature (see, e.g., [4]–[10], [12], [13], [16], [17], [19], [23], [27], [32], [37], [40], [45], and the references therein). On the other hand, the classic problem of absolute stability of a class of nonlinear control systems with a fixed matrix in the linear part of the system and one or multiple uncertain nonlinearities satisfying the sector constraints has been extensively studied [3], [20], [26], [29]–[31], [34], [38], [42], [44], [46] long before the publication of the initial work of Kharitonov [22] which laid foundation for the problem of robust stability. Meanwhile, in the light of modern robustness terminology, absolute stability can be considered as the robust global asymptotic stability with respect to variations (or changes) of nonlinearities from a given class.

Recently, there has been some work devoted to the investigation of the more general problem of *robust absolute stability* of nonlinear control systems with uncertainties both in the linear part and in the nonlinear part of the system [2], [7], [14], [15], [41], [43]. To the best of our knowledge, in all papers on this problem, only sufficient conditions of robust absolute stability were obtained based mainly on the well-known circle and Popov criteria. Thus, at present, the problem of obtaining *necessary and sufficient* conditions of robust absolute stability is of great theoretical and practical interest.

The main purpose of this paper is to establish necessary and sufficient conditions of *robust absolute stability* for a class of nonlinear discrete-time systems with time-varying matrix uncertainty of polyhedral type and multiple time-varying sector nonlinearities. Using the variational method developed in [29] and [38] for the problem of absolute stability of nonlinear discrete-time systems and the discrete version of Lyapunov second method [24], [44], we establish criteria for the robust absolute stability of the class of systems under consideration. The parametric classes of Lyapunov functions which define the necessary and sufficient conditions of robust absolute stability of such systems are identified. An algebraic criterion for robust absolute stability in the form of solvability conditions of a set of matrix equations is obtained using Lyapunov functions from a class of piecewise-linear functions of the vector norm type.

In general, the main problem related to the implementation of the obtained criteria of robust absolute stability is their computational complexity [35]. Therefore, following [4], [32], [39], and [45], we obtain several simple and computationally efficient sufficient conditions for robust absolute stability of the class of discrete-time systems considered herein. We will indicate that these conditions become necessary and sufficient for a few special cases. We will conclude the paper with an example of application of these results to a special class of systems with time-varying interval matrices in the linear part.

II. Problem Statement and Preliminaries

Let R^n denote real $n$space. If $x \in R^n$, then $x^T = [x_1, \ldots, x_n]$ denotes the transpose of x. Let $R^{m\times n}$ denote the set of $m \times n$ real matrices. If $A = [a_{ij}] \in R^{m\times n}$, then A^T denotes the transpose of A. We let $\|x\|$ denote any one of the equivalent vector norms on R^n. In particular, the l_p norms $\|x\|_p$, $1 \le p \le \infty$, are defined by

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}, \qquad 1 \le p < \infty$$

and

$$\|x\|_\infty = \max_{1\le i\le n} |x_i|.$$

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The matrix norm $||A||$, defined on $R^{n\times n}$, and induced by the vector norm $||x||$ in R^n, are defined as

$$||A|| = \max_{||x||=1} ||Ax||.$$

In particular, we have

$$||A||_\infty = \max_{||x||_\infty=1} ||Ax||_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|.$$

We consider the following class of nonlinear discrete-time systems described by the equations

$$x(k+1) = A(k)x(k) + \sum_{j=1}^{r} b_j\phi_j(\sigma(k), k), \qquad k = 0, 1, 2, \ldots \tag{1}$$

where $x(k) \in R^n$, $A(k) \in R^{n\times n}$, $\sigma^T(k) = [\sigma_1(k), \ldots, \sigma_r(k)]$ with $\sigma_j(k) = c_j^T x(k)$, and $b_j \in R^n$ and $c_j \in R^n$ are constant vectors for $j = 1, \ldots, r$. The nonlinear functions $\phi_j(\sigma(k), k)$, $j = 1, \ldots, r$, are defined below. It is assumed that for each $k \in \{0, 1, 2\cdots\}$, the time-varying matrix $A(k)$ in (1) is chosen arbitrarily from a given polytope of matrices

$$\mathcal{A} = co\{A_1, \ldots, A_q\} \subset R^{n\times n} \tag{2}$$

where $co\{\cdot\}$ denote the convex hull of a set. The matrix polytope $\mathcal{A}$ describes structured parametric uncertainty in the linear part of (1). We assume that the uncertain time-varying nonlinear functions $\phi_j(\sigma(k), k)$, $j = 1, \ldots, r$, are defined for any $x(k) \in R^n$ and satisfy the conditions $\phi_j(0, k) \equiv 0$, $j = 1, \ldots, r; k = 0, 1, 2, \ldots$, and we assume the sector constraints

$$\alpha_j\sigma_j^2(k) \le \phi_j(\sigma(k), k)\sigma_j(k) \le \beta_j\sigma_j^2(k), \qquad j = 1, \ldots, r \tag{3}$$

where $\alpha_j \in R$ and $\beta_j \in R$, $j = 1, \ldots, r$, are given constants. We use Φ to denote the set of all such time-varying nonlinear vector-functions $\phi(\sigma(k), k)$, where

$$\phi^T(\sigma(k), k) = [\phi_1(\sigma(k), k), \ldots, \phi_r(\sigma(k), k)].$$

Thus, any solution $x_{A,\phi}(k, k_0, x_0)$ of (1) is defined by an arbitrary choice of a time-varying matrix $A(k)$ from the set $\mathcal{A}$ and a vector nonlinearity $\phi(\sigma(k), k)$ from the set Φ, in addition to an initial state (k_0, x_0).

Note that due to the fact that $\phi(0, k) \equiv 0$, we have

$$x_{A,\phi}(k, k_0, 0) \equiv 0, \qquad k = k_0, k_0+1, \ldots$$

for any matrix sequence $\{A(k) \in \mathcal{A}, k = k_0, k_0+1, \ldots\}$ and any nonlinearity $\phi(\sigma(k), k) \in \Phi$. Therefore, we will use the notation $x(k) \equiv 0$ for the zero solution of (1).

Similar to the definitions given in [15], [43], in this paper, the robust absolute stability of (1) will be considered in the sense of the following definition.

Definition 2.1: The system (1) is said to be *robustly absolutely stable* with respect to the set $\mathcal{A}\times\Phi$ defined by (2) and (3) if its zero solution $x(k) \equiv 0$ is globally asymptotically stable for any time-varying matrix $A(k) \in \mathcal{A}$ and any vector nonlinearity $\phi(\sigma(k), k) \in \Phi$. ∎

We note that if there is no uncertainty in the linear part of (1), that is, if all matrices A_i, $i = 1, \ldots, q$, in (2) are identical ($A_i \equiv A$, $i = 1, \ldots, q$) and the set $\mathcal{A}$ degenerates to the "singleton" or "point" A in the matrix space $R^{n\times n}$, then the problem of robust absolute stability becomes that of absolute stability of (1) with respect to the set Φ of time-varying vector nonlinearities $\phi(\sigma(k), k)$ defined by (3), which was previously discussed by many authors [3], [20], [29], [30], [38], [42], [44], [46]. On the other hand, if $\phi(\sigma(k), k) \equiv 0$, that is in accordance with (3), if $\alpha_j = \beta_j = 0$, for all $j = 1, \ldots, r$, the problem of robust absolute stability reduces to that of robust stability of the linear time-varying system

$$x(k+1) = A(k)x(k), \qquad k = 0, 1, 2, \ldots \tag{4}$$

with respect to the set $\mathcal{A}$ defined by (2), which was also considered in many previous works (see [5], [6], [8], [10], [16], [17], [23], [27], [32], [37], [40], [45], and the references in [12] and [19]).

The main goal of this work is to obtain necessary and sufficient conditions for robust absolute stability of (1) with respect to the set $\mathcal{A}\times\Phi$ defined by (2) and (3). Our main results are given in Sections IV and V.

III. Reduction to the Problem of Robust Stability

In this section, we consider along with nonlinear system (1) the linear time-varying system

$$x(k+1) = \left(A(k) + \sum_{j=1}^{r} \mu_j(k)b_jc_j^T\right)x(k), \qquad k = 0, 1, 2, \ldots \tag{5}$$

where $A(k) \in \mathcal{A}$ and $\mu_j(k)$, $j = 1, \ldots, r$, are arbitrary functions satisfying the inequalities

$$\alpha_j \le \mu_j(k) \le \beta_j, \qquad j = 1, \ldots, r \tag{6}$$

for all $k = 0, 1, 2, \ldots$. The set of such vector-function $\mu(k)$, where $\mu^T(k) = [\mu_1(k), \ldots, \mu_r(k)]$, will be denoted by M.

The system (5) can be obtained from (1) by considering the functions $\phi_j(\sigma(k), k)$ of the particular type given by $\phi_j(\sigma(k), k) = \mu_j(k)\sigma_j(k)$, $j = 1, \ldots, r$, which form a subset of Φ. In this case, the inequalities (6) are a direct consequence of inequalities (3). *Robust absolute stability* of (5) with respect to the set $\mathcal{A}\times M$ will be understood in the sense of Definition 2.1 given in Section II, replacing Φ by M and $\phi(\sigma(k), k) \in \Phi$ by $\mu(k) \in M$.

Following [8], [16], and [30], it is easy to show that the nonlinear system (1) on the set $\mathcal{A}\times\Phi$ and the linear system (5) on the set $\mathcal{A}\times M$ are equivalent to the following time-invariant difference inclusion:

$$x(k+1) \in F(x(k)), \qquad k = 0, 1, 2, \ldots \tag{7}$$

where the multivalued vector-function $F(x)$ is defined for all $x \in R^n$ by

$$F(x) = \left\{ y\colon y = \left(\sum_{i=1}^{q}\lambda_iA_i + \sum_{j=1}^{r}\gamma_jb_jc_j^T\right)x, \quad \lambda_i \ge 0;\quad i = 1, \ldots, q;\quad \sum_{j=1}^{q}\lambda_i = 1; \quad \alpha_j \le \gamma_j \le \beta_j,\ j = 1, \ldots, r \right\}. \tag{8}$$

The equivalence is regarded in the sense of coincidence of the sets of solutions of (1) [for all admissible $A(k) \in \mathcal{A}$, $\phi(\sigma(k), k) \in \Phi$], of (5) [for all admissible $A(k) \in \mathcal{A}$, $\mu(k) \in M$], and of the difference inclusion (7) and (8). Therefore, the problem of robust absolute stability of the nonlinear system (1) with respect to the set $\mathcal{A} \times \Phi$ is equivalent to a similar problem for the linear system (5) with respect to the set $\mathcal{A} \times M$, and both problems can be reduced to the problem of global asymptotic stability of the zero solution $x(k) \equiv 0$ of the difference inclusion (7) and (8).

As a result of the preceding discussions, we obtain the following lemma.

Lemma 3.1: For the robust absolute stability of (1) with respect to the set $\mathcal{A} \times \Phi$, it is necessary and sufficient that (5) be robustly absolutely stable with respect to the set $\mathcal{A} \times M$. ■

We introduce into consideration the matrices

$$\tilde{A}_{i\nu} = A_i + \sum_{j=1}^{r} \tilde{\gamma}_{\nu j} b_j c_j^T, \qquad i = 1, \ldots, q; \quad \nu = 1, \ldots, s; \quad s = 2^r \tag{9}$$

where the parameters $\tilde{\gamma}_{\nu j}$ can independently take only the extreme values $\tilde{\gamma}_{\nu j} = \alpha_j$ or $\tilde{\gamma}_{\nu j} = \beta_j$, for $j = 1, \ldots, r$. It can easily be seen that any vector $y \in F(x)$ in (8) admits an equivalent representation as $y = Ax$, where the matrix $A \in R^{n \times n}$ belongs to the matrix polytope

$$\tilde{\mathcal{A}} = co\left\{\tilde{A}_{11}, \ldots, \tilde{A}_{1s}; \ldots; \tilde{A}_{q1}, \ldots, \tilde{A}_{qs}\right\} \tag{10}$$

which is the convex hull of the matrices $\tilde{A}_{i\nu}$, $i = 1, \ldots, q$; $\nu = 1, \ldots, s$; $s = 2^r$, defined by (9). Therefore, in accordance with Lemma 3.1, the problem of robust absolute stability of (1) with respect to the set $\mathcal{A} \times \Phi$ reduces to an equivalent problem of robust stability of the linear time-varying system (4) with respect to the set $\tilde{\mathcal{A}}$ defined by (9) and (10) in the sense of the following definition [5], [6].

Definition 3.1: The system (4) is said to be *robustly stable* with respect to the set $\tilde{\mathcal{A}}$ if the zero solution $x(k) \equiv 0$ of this system is globally asymptotically stable for any time-varying matrix $A(k) \in \tilde{\mathcal{A}}$. ■

Note that for linear system (4) the concept of *global* asymptotic stability is equivalent to that of *local* asymptotic stability [24]. For the same reason, robust stability of (4) with respect to the set $\tilde{\mathcal{A}}$ is equivalent to the condition

$$\lim_{k\to\infty} \|x_A(k, k_0, x_0)\| = 0 \tag{11}$$

for any solution $x_A(k, k_0, x_0)$ of (4) corresponding to any $k_0 \in \{0, 1, 2, \ldots\}$, any initial state $x(k_0) = x_0 \in R^n$, and arbitrary choice of the matrix sequence $\{A(k) \in \tilde{\mathcal{A}}, k = k_0, k_0+1, k_0+2, \ldots\}$.

Note also that, it follows from the results obtained in [8], [11], if the limit condition (11) is valid, then there exist constants $L \geq 1$ and $0 < \xi < 1$ such that the estimate

$$\|x_A(k, k_0, x_0)\| \leq L\|x_0\|\xi^{k-k_0}, \qquad k = k_0, k_0+1, \ldots \tag{12}$$

is fulfilled for any $k_0 \in \{0, 1, 2, \ldots\}$, any $x_0 \in R^n$ and any matrix sequence $\{A(k) \in \tilde{\mathcal{A}}, k = k_0, k_0+1, \ldots\}$. The opposite assertion is obvious. Thus, the following statement holds.

Lemma 3.2: The system (1) is robustly absolutely stable with respect to the set $\mathcal{A} \times \Phi$ if and only if (4) is globally exponentially stable [in the sense of the inequality (12)] with respect to the set $\tilde{\mathcal{A}}$ defined by (9) and (10). ■

Lemma 3.2 is the starting point for obtaining necessary and sufficient conditions of robust absolute stability presented in the next section.

Using the property that for any compact set $\tilde{\mathcal{A}} \in R^{n\times n}$ the difference inclusions (7) and

$$x(k+1) \in coF(x(k))$$

where $F(x) = \{y\colon y = Ax, A \in \tilde{\mathcal{A}}\}$, are asymptotic stable simultaneously (see, e.g., [8, Proposition 3.2]); the well-known fact that the convex compact set can be approximated with any accuracy by convex polyhedron; and [11, Th. 6 and Corollary 7], we can show that robust absolute stability of (1) with respect to the set $\mathcal{A} \times \Phi$, where $\mathcal{A}$ is an arbitrary compact set in $R^{n\times n}$, is equivalent to robust absolute stability of (1) with respect to some approximating set $co\{A_1, \ldots, A_q\} \times \Phi$. In this sense, the general case of a compact set $\mathcal{A}$ for (1) is reduced to the case of a convex polytope. For this reason, we can restrict our analysis in the present paper to the case of a convex polytope $\mathcal{A}$ defined by (2) for (1).

IV. Main Results

In this section, we will derive necessary and sufficient conditions for robust absolute stability of (1) with respect to the set $\mathcal{A} \times \Phi$. We will employ the variational method [6], [38] and the discrete version of Lyapunov second method [24], [44], which were also used in [29] and [30] for the problem of absolute stability of nonlinear discrete-time control systems with fixed matrix in the linear part [i.e., when $A_i \equiv A$, $i = 1, \ldots, q$, in (2)].

First, we present a criterion for robust absolute stability that can be obtained by the variational method. Following [6], we use Π_k to denote the set of all matrix products $\pi_k(i[k], \nu[k])$ of vertex matrices $\tilde{A}_{i\nu}$, $i = 1, \ldots, q$; $\nu = 1, \ldots, s$; $s = 2^r$, defined by (9), of length k

$$\pi_k(i[k], \nu[k]) = \tilde{A}_{i_k\nu_k} \ldots \tilde{A}_{i_1\nu_1} \tag{13}$$

where

$$i[k] = (i_1, \ldots, i_k) \in \{1, \ldots, q\}^k$$
$$\nu[k] = (\nu_1, \ldots, \nu_k) \in \{1, \ldots, s\}^k$$

$s = 2^r$, and $\{1, \ldots, q\}^k$ and $\{1, \ldots, s\}^k$ are the kth cross products of the sets $\{1, \ldots, q\}$ and $\{1, \ldots, s\}$, respectively. It is assumed that $\pi_0 = I_n$, where I_n is an $n \times n$ identity matrix. Obviously, the set Π_k contains $q^k s^k$ matrix products $\pi_k(i[k], \nu[k])$ of the form (13). For brevity, we will use sometimes the notation π_k for matrices $\pi_k(i[k], \nu[k]) \in \Pi_k$.

We are now in a position to establish the following result.

Theorem 4.1: The system (1) is robustly absolutely stable with respect to the set $\mathcal{A} \times \Phi$ if and only if there exists a finite integer $\hat{k} \geq 1$, such that

$$\left\|\pi_{\hat{k}}(i[\hat{k}],\, \nu[\hat{k}])\right\| < 1, \qquad \text{for all } \pi_{\hat{k}}(i[\hat{k}],\, \nu[\hat{k}]) \in \Pi_{\hat{k}}. \quad (14)$$

The proof of Theorem 4.1 is similar to the proof of [29, Lemma A.1] with some minor modifications, and therefore, is omitted here. Note that the statement of Theorem 4.1 can also be considered as a corollary of the part 2 of [8, Theorem 3.1 and Proposition 3.2].

From Theorem 4.1, the following corollary can easily be obtained.

Corollary 4.1: The system (1) is robustly absolutely stable with respect to the set $\mathcal{A} \times \Phi$ if there exists in R^n a vector norm $\|\cdot\|$ such that the induced matrix norm $\|\tilde{A}_{i\nu}\| < 1$ for any matrix $\tilde{A}_{i\nu}$ defined by (9), $i = 1, \ldots, q$; $\nu = 1, \ldots, s$; $s = 2^r$. ■

Corollary 4.1 is obvious because

$$\Pi_1 = \left\{\tilde{A}_{11}, \ldots, \tilde{A}_{1s}; \ldots; \tilde{A}_{q1} \ldots, \tilde{A}_{qs}\right\}$$

and under the above condition, we will have the inequality (14) with $\hat{k} = 1$.

Let

$$\rho(A) = \max_{1 \leq i \leq n} |\lambda_i(A)|$$

denote the spectral radius of matrix A, where $\lambda_i(A)$, $i = 1, \ldots, n$, are eigenvalues of the matrix A. Recall that an $n \times n$ matrix A is said to be Schur stable if $\rho(A) < 1$. It is well known that the condition $\rho(A) < 1$ is equivalent to the existence of a finite integer $\hat{k} \geq 1$, such that $\|A^{\hat{k}}\| < 1$. Since $(\tilde{A}_{i\nu})^{\hat{k}} \in \Pi_{\hat{k}}$ for any matrix $\tilde{A}_{i\nu}$ of the form (9), it follows from (14) that $\|(\tilde{A}_{i\nu})^{\hat{k}}\| < 1$. As a result, we obtain the following corollary.

Corollary 4.2: For the robust absolute stability of (1) with respect to the set $\mathcal{A} \times \Phi$, it is necessary that every vertex matrix $\tilde{A}_{i\nu}$ defined by (9) is Schur stable. ■

The next two corollaries represent special cases when Schur stability of the vertex matrices $\tilde{A}_{i\nu}$ is not only necessary but also sufficient for robust absolute stability of (1). Other special cases are given in Theorem 5.2 in Section V.

Corollary 4.3: If the vertex matrices $\tilde{A}_{i\nu}$, $i = 1, \ldots, q$; $\nu = 1, \ldots, s$; $s = 2^r$ defined by (9) are pairwise commutative, i.e., if $\tilde{A}_{i\nu}\tilde{A}_{jl} = \tilde{A}_{jl}\tilde{A}_{i\nu}$ for all $i, j = 1, \ldots, q$; $\nu, l = 1, \ldots, s$; then for robust absolute stability of (1) with respect to the set $\mathcal{A} \times \Phi$, it is necessary and sufficient that the matrices $\tilde{A}_{i\nu}$, $i = 1, \ldots, q$; $\nu = 1, \ldots, s$; $s = 2^r$, are Schur stable.

Proof: Necessity follows from Corollary 4.2. For the proof of sufficiency, let us note that in the case of pairwise commutative matrices $\tilde{A}_{i\nu}$, $i = 1, \ldots, q$; $\nu = 1, \ldots, s$; $s = 2^r$, any matrix $\pi_k \in \Pi_k$ can be represented in the form

$$\pi_k = \tilde{A}_{qs}^{k_{qs}} \ldots \tilde{A}_{q1}^{k_{q1}} \ldots \tilde{A}_{1s}^{k_{1s}} \ldots \tilde{A}_{11}^{k_{11}} \quad (15)$$

where

$$\sum_{i=1}^{q} \sum_{\nu=1}^{s} k_{i\nu} = k.$$

Since

$$\max_{1 \leq i \leq q,\, 1 \leq \nu \leq s} \rho(\tilde{A}_{i\nu}) < 1$$

for any induced matrix norm there exist constants $L \geq 1$ and $0 < \xi < 1$ such that

$$\max_{1 \leq i \leq q,\, 1 \leq \nu \leq s} \left\|\left(\tilde{A}_{i\nu}\right)^k\right\| \leq L\xi^k, \qquad k = 0, 1, 2, \ldots.$$

Therefore, using the last estimate, we have from (15) the inequality

$$\|\pi_k\| \leq L^{qs}\xi^k, \qquad \text{for all } \pi_k \in \Pi_k;\ k = 0, 1, 2, \ldots. \quad (16)$$

By choosing the integer $\hat{k} > qs \ln L / |\ln \xi|$, we obtain from (16) the inequality (14), which completes the proof of Corollary 4.3. ■

We note that Corollary 4.3 follows also from the discrete-time counterpart of [25, Th. 2].

Note that sufficiency in Corollary 4.3 can also be proved by the Lyapunov second method [24]. As is shown in [33], in this case, the linear time-varying system (4) has a common quadratic Lyapunov function for any time-varying matrix $A(k) \in \tilde{\mathcal{A}}$.

To formulate the next corollary, we recall that a real matrix $A \in R^{n \times n}$ is said to be normal if $A^T A = AA^T$ [18], [28]. If $A \in R^{n \times n}$ is normal, then $\|A\|_2 = \rho(A)$, where

$$\|A\|_2 = \max_{\|x\|_2 = 1} \|Ax\|_2 = \sqrt{\rho(A^T A)}$$

is the spectral norm of the matrix A and

$$\|x\|_2 = \sqrt{x^T x} = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$$

is the Euclidean vector norm (cf. [18] and [28]). Therefore, using this property and Corollaries 4.1 and 4.2, we obtain the following corollary.

Corollary 4.4: If every vertex matrix $\tilde{A}_{i\nu}$, $i = 1, \ldots, q$; $\nu = 1, \ldots s$; $s = 2^r$, defined by (9) is normal, then for robust absolute stability of (1) with respect to the set $\mathcal{A} \times \Phi$, it is necessary and sufficient that the matrices $\tilde{A}_{i\nu}$, $i = 1, \ldots, q$; $\nu = 1, \ldots s$; $s = 2^r$, are Schur stable. ■

Note that the conditions of Corollary 4.4 assure the existence of the common Lyapunov function $V(x) = (\|x\|_2)^2 = x^T x$ for (4) with any time-varying matrix $A(k) \in \tilde{\mathcal{A}}$.

Theorem 4.1 forms the basis for a computer-aided test of robust absolute stability. Based on this result, we can develop a numerical algorithm, analogous to that in [6], which, for a robustly absolutely stable system (1), is theoretically capable of determining robust absolute stability in a finite number of steps. However, since in reality the condition (14) may be fulfilled for extremely large $\hat{k}$, this algorithm can not always be implemented due to the restrictions of computational capacity. Therefore, this computer-aided test usually provides only sufficient conditions for robust absolute stability. For this reason it is very important to obtain other necessary and sufficient conditions of robust absolute stability which could be less involved in actual implementations.

By using the discrete-time version of the Lyapunov second method [24], [44], and its applications in [9], [10], [16], and [30] to the problem of absolute stability of nonlinear discrete-time systems with fixed linear part and to the problem of robust stability of uncertain discrete-time systems, one can establish necessary and sufficient conditions of robust absolute stability of (1) in the form similar to [30, Th. 3–5] and corresponding results in [9], [10], and [16]. We note that the present results are mostly concerned with identifying the parametric classes of Lyapunov functions defining the necessary and sufficient conditions of robust absolute stability of (1).

The main result of this paper obtained in the framework of this approach is stated in the following theorem.

Theorem 4.2: For the robust absolute stability of (1) with respect to the set $\mathcal{A} \times \Phi$, it is necessary and sufficient that for some integer $m \geq n$ there exists a full column rank matrix $H \in R^{m \times n}$ and a constant θ $(0 < \theta < 1)$ such that a piecewise-linear Lyapunov function of the polyhedral vector norm type

$$V_H(x) = \|x\|_H = \|Hx\|_\infty \tag{17}$$

satisfies the inequality

$$\max_{y \in F(x)} V_H(y) \leq \theta V_H(x), \qquad x \in R^n \tag{18}$$

where the set $F(x) \subset R^n$ is defined at each point $x \in R^n$ by (8). ■

The proof of Theorem 4.2 follows similar steps as the proof of Theorem 3 given in [30] for absolute stability with some nonessential modifications, and is therefore omitted here due to space limitations.

We note that the positive definiteness of the polyhedral Lyapunov function $V_H(x)$ in Theorem 4.2 follows directly from the rank condition on the matrix H. The level surfaces of this function are boundaries of centrally symmetric convex polytope, and each row H_i, $i = 1, \ldots, m$, of the matrix H specify the normals to the faces of the convex polytope. Note also that the inequality (18) guarantees the strict decreasing of the Lyapunov function $V_H(x)$ along the solutions of (1), and the constant θ characterizes the rate of decrease.

From Theorem 4.2 and Corollary 4.1, the following corollary can be obtained.

Corollary 4.5: The system (1) is robustly absolutely stable with respect to the set $\mathcal{A} \times \Phi$, if and only if there exists in R^n a polyhedral vector norm $\|x\|_H$ of the type (17), such that the corresponding induced matrix norm satisfies the condition

$$\|\tilde{A}_{i\nu}\|_H < 1, \qquad i = 1, \ldots, q; \quad \nu = 1, \ldots s; \quad s = 2^r. \tag{19}$$

Theorem 4.2 makes it also possible to consider Lyapunov functions of the type of even degree $2p$ homogeneous polynomial (see, e.g., [10], [30], and [31])

$$V_{H,p}(x) = \sum_{i=1}^{m} (H_i x)^{2p} = (\|Hx\|_{2p})^{2p} \tag{20}$$

where H_i, $i = 1, \ldots, m$, is ith row of the matrix H in (17). Under the above-mentioned rank condition the function, $V_{H,p}(x)$ will be a strictly convex function in the space R^n.

We have the following corollary.

Corollary 4.6: For the robust absolute stability of (1) with respect to the set $\mathcal{A} \times \Phi$, it is necessary and sufficient that there exists a full column rank matrix $H \in R^{m \times n}$ (with H_i being its ith row, $i = 1, \ldots, m$), a finite integer $p \geq 1$ and a constant θ $(0 < \theta < 1)$, such that the Lyapunov function $V_{H,p}(x)$ defined by (20) satisfies the inequality

$$\max_{y \in F(x)} V_{H,p}(y) \leq \theta V_{H,p}(x), \qquad x \in R^n.$$

■

The proof of Corollary 4.6 is similar to [30, proof of Th. 4] and is omitted here.

One of the main advantages of the Lyapunov function $V_{H,p}(x)$ of the form (20) in comparison with the function $V_H(x)$ of the form (17) is that it is a smooth function everywhere in R^n. It gives the opportunity of using a well developed technique of smooth optimization for numerical construction of such functions.

The use of Theorem 4.2 enables us to obtain a criterion for robust absolute stability of (1) in algebraic form. Such algebraic criterion of robust absolute stability is stated in the next theorem.

Theorem 4.3: For the robust absolute stability of (1) with respect to the set $\mathcal{A} \times \Phi$, it is necessary and sufficient that there exists a finite integer $m \geq n$, a full column rank matrix $H \in R^{m \times n}$, and $m \times m$ matrices $\Gamma_{i\nu}$, $i = 1, \ldots, q$; $\nu = 1, \ldots, s$; $s = 2^r$, satisfying the conditions $\|\Gamma_{i\nu}\|_\infty < 1$, $i = 1, \ldots, q$; $\nu = 1, \ldots, s$; $s = 2^r$, such that the matrix relations

$$H\tilde{A}_{i\nu} = \Gamma_{i\nu} H, \qquad i = 1, \ldots, q; \quad \nu = 1, \ldots, s; \quad s = 2^r \tag{21}$$

are satisfied. ■

The proof of Theorem 4.3 follows with some nonessential modifications the proof of an analogous criterion in [30, Th. 5] establishing the necessary and sufficient conditions of absolute stability, and is omitted here.

Note that the conditions of Theorem 4.3 are automatically satisfied if $\|\tilde{A}_{i\nu}\|_\infty < 1$ for any matrix $\tilde{A}_{i\nu}$, $i = 1, \ldots, q$; $\nu = 1, \ldots, s$; $s = 2^r$. In this case, $H = I_n$, $\Gamma_{i\nu} = \tilde{A}_{i\nu}$, $i = 1, \ldots, q$; $\nu = 1, \ldots, s$; $s = 2^r$, and $V_H(x) = \|x\|_\infty$ (see also Corollaries 4.1 and 4.5).

In connection with Theorem 4.3 we also note that, in fact, the matrix relations (21) are equivalent to the conditions (19) for the matrix norm induced by the Lyapunov function $V_H(x)$ in (17), which defines a polyhedral vector norm $\|x\|_H$. The relations (21) can be understood as the conditions for the generalized similarity of the vertex $n \times n$ matrices $\tilde{A}_{i\nu} \in R^{n \times n}$, $i = 1, \ldots, q$; $\nu = 1, \ldots, s$; $s = 2^r$, which define the set $\tilde{\mathcal{A}} \subset R^{n \times n}$ for (4), to the $m \times m$ matrices $\Gamma_{i\nu} \in R^{m \times m}$, $i = 1, \ldots, q$; $\nu = 1, \ldots, s$; $s = 2^r$, which define the following time-varying linear system:

$$z(k+1) = \Gamma(k) z(k), \qquad k = 0, 1, 2, \ldots \tag{22}$$

where $z(k) \in R^m$ and

$$\Gamma(k) \in co\{\Gamma_{11}, \ldots, \Gamma_{1s}, \ldots, \Gamma_{q1}, \ldots, \Gamma_{qs}\} \subset R^{m \times m}.$$

The matrix H in (21) plays the role of a generalized transformation matrix. The matrix relations (21) establish the existence of a linear transformation $z = Hx$, which connects (4) on the set $\tilde{\mathcal{A}}$ and (22) on the set $co\{\Gamma_{11}, \ldots, \Gamma_{1s}; \ldots; \Gamma_{q1}, \ldots, \Gamma_{qs}\}$.

We note that (22) is robustly stable with respect to the set

$$co\{\Gamma_{11}, \ldots, \Gamma_{1s}, \ldots, \Gamma_{q1}, \ldots, \Gamma_{qs}\} \subset R^{m \times m}$$

since $\|\Gamma_{i\nu}\|_\infty < 1$, $i = 1, \ldots, q$; $\nu = 1, \ldots, s$; $s = 2^r$. Moreover, we can choose θ in (18) as

$$\theta = \max_{1\le i\le q,\, 1\le \nu\le s} \{\|\Gamma_{i\nu}\|_\infty\}.$$

Thus, Theorem 4.2 reduces the problem of robust absolute stability for (1) with respect to the set $\mathcal{A} \times \Phi$ to the problem of existence of a polyhedral vector norm type Lyapunov function (17). In turn, Theorem 4.3 reduces the problem of the construction of such a Lyapunov function to the problem of solvability of the matrix relations (21), which can be considered as a set of matrix equations in terms of unknown matrices $H \in R^{m\times n}$ and $\Gamma_{i\nu} \in R^{m\times m}$, $i = 1, \ldots, q$; $\nu = 1, \ldots, s$; $s = 2^r$.

In general, the problem of solving matrix equations (21), and consequently, the problem of robust absolute stability for (1) can effectively be solved only by numerical methods, i.e., with the use of computers. Unfortunately, the situation is complicated by the fact that Theorems 4.2 and 4.3 say nothing about how large an integer m should be to satisfy the conditions of these theorems. In principle, the number m can be much larger than its lower bound n, and it should be chosen experimentally in practical numerical calculations by incrementing its value. If the value of m is fixed (for example, $m = n$), then the conditions of Theorems 4.2 and 4.3 become only sufficient for robust absolute stability of (1). We note that an efficient numerical algorithm was proposed in [36] for solving matrix equations similar to (21) with the use of linear programming and the idea of scaling; the algorithm of [36] can also be used for checking the conditions of Theorems 4.2 and 4.3. The improvement of these sufficient conditions can be achieved only at the expense of increasing the integer parameter $m > n$.

Nevertheless, for (1), simpler sufficient conditions of robust absolute stability, which often coincide with necessary and sufficient conditions, exist in special cases. Some classes of these systems are discussed in Section V.

In conclusion of this section, we make a general remark concerning the necessary and sufficient conditions of robust absolute stability established in Theorems 4.1–4.3 above.

As noted in Section II, if there is no uncertainty in the linear part of (1), that is, if all matrices A_i in (2) are identical ($A_i = A =$ constant matrix, $i = 1, \ldots, q$), then the problem of robust absolute stability with respect to the set $\mathcal{A} \times \Phi$ turns into the classic problem of absolute stability with respect to the set Φ of *time-varying* vector nonlinearities $\phi(\sigma(k), k)$ that satisfy sector constraints (3).

In [3], it is shown (cf. [3, Theorem 3]) that for absolute stability of discrete-time systems of type (1) (with $A_i = A =$ constant matrix, $i = 1, \ldots, q$) in the class Φ it is necessary and sufficient that they are absolutely stable in the subclass $\overline{\Phi} \subset \Phi$ of *time-invariant* nonlinearities $\phi(\sigma(k)) \in \overline{\Phi}$ that do not depend *explicitly* on the discrete time $k \in \{0, 1, 2, \ldots\}$ [i.e., depend only on $\sigma(k)$] and satisfy the same sector constraints (3). Therefore, in the case when $A_i = A =$ constant matrix, $i = 1, \ldots, q$, Theorems 4.1–4.3 of this paper give the necessary and sufficient conditions of absolute stability both for time-varying Lur'e system (1) in the class Φ and for time-invariant Lur'e system (1) in the class $\overline{\Phi} \subset \Phi$.

On the other hand, the well-known principal result for absolute stability of discrete-time control systems both in the class Φ of time-varying nonlinearities $\phi(\sigma(k), k)$ and in the class $\overline{\Phi}$ of time-invariant nonlinearities $\phi(\sigma(k))$ that satisfy sector conditions (3) is Tsypkin's frequency criterion [42] in the case of a single nonlinearity ($r = 1$) and its generalizations [20], [46] to the case of several nonlinearities ($r > 1$) in (1) (i.e., the so-called circle criterion). The example given in [29], using the conditions similar to (14) in Theorem 4.1, shows that Tsypkin's criterion does not yield necessary and sufficient conditions for absolute stability in general case. Therefore, the results of the present paper include the necessary and sufficient conditions of absolute stability for discrete-time Lur'e control systems which give opportunity to obtain wider domains of absolute stability in parameter space of the system under consideration than sufficient conditions of absolute stability given by Tsypkin's criterion and the circle criterion.

V. Some Sufficient Conditions for Robust Absolute Stability

In this section, we establish several simple sufficient conditions for robust absolute stability of (1) with respect to the set $\mathcal{A} \times \Phi$. The proposed conditions are special corollaries of the results established in Section IV, and are relatively simple to implement, because these conditions are reduced to checking the Schur stability of some particular test matrix. It is shown that these conditions become necessary and sufficient for several special cases of (1).

Let $|A|$ denote the matrix obtained from A by taking the absolute value of all entries, i.e., if $A = [a_{ij}] \in R^{m\times n}$, then $|A| = [|a_{ij}|] \in R^{m\times n}$. For two $m \times n$ matrices, $A = [a_{ij}] \in R^{m\times n}$ and $B = [b_{ij}] \in R^{m\times n}$, $A \le B$ ($A < B$) denotes an element-wise inequality, i.e., $a_{ij} \le b_{ij}$ ($a_{ij} < b_{ij}$), $i = 1, \ldots, m$; $j = 1, \ldots, n$. The Hadamard product of A and B is written as $A \circ B$ and defined in [18], [32] by the element-wise product $A \circ B = [a_{ij}b_{ij}] \in R^{m\times n}$. In the inequality $A \ge 0$, we use 0 to denote the matrix of appropriate dimension whose entries are all equal to zero.

Let us associate with a given set of matrices $\tilde{A}_{i\nu}$, $i = 1, \ldots, q$; $\nu = 1, \ldots, s$; $s = 2^r$, defined by (9), a nonnegative majorant matrix

$$\hat{A} = \max_{1\le i\le q,\, 1\le \nu\le s} \left\{ \left|\tilde{A}_{i\nu}\right| \right\} \tag{23}$$

where the maximum is understood to be element-wise.

The following theorem can now be established.

Theorem 5.1: The system (1) is robustly absolutely stable with respect to the set $\mathcal{A} \times \Phi$, if the matrix $\hat{A}$ in (23) is Schur stable, i.e., $\rho(\hat{A}) < 1$.

Proof: Since $\hat{A} \ge 0$ and $\rho(\hat{A}) < 1$, then from [21, Lemma 3.1] it follows that there exists a positive-definite diagonal matrix $H \in R^{n\times n}$

$$H = \operatorname{diag}\{h_1, \ldots, h_n\} \quad h_i > 0, \qquad i = 1, \ldots, n \tag{24}$$

such that $\|H\hat{A}H^{-1}\|_\infty < 1$. Clearly, rank$(H) = n$. Let us denote $\Gamma = H\hat{A}H^{-1}$, $\|\Gamma\|_\infty < 1$. Obviously, $H\hat{A} = \Gamma H$.

It is easy to show that there exist matrices $W_{i\nu} \in R^{n\times n}$ such that $\tilde{A}_{i\nu} = \hat{A} \circ W_{i\nu}$ and $|W_{i\nu}| \le E_n$, $i = 1, \ldots, q$;

$\nu = 1, \ldots, s$; $s = 2^r$, where we use $E_n \in R^{n\times n}$ to denote the $n \times n$ matrix whose entries are all equal to one. We have

$$H\tilde{A}_{i\nu} = H\hat{A} \circ W_{i\nu} = W_{i\nu} \circ H\hat{A} = W_{i\nu} \circ \Gamma H = \Gamma_{i\nu} H$$

where $\Gamma_{i\nu} = W_{i\nu} \circ \Gamma$, $i = 1, \ldots, q$; $\nu = 1, \ldots, s$; $s = 2^r$, and $\|\Gamma_{i\nu}\|_\infty \le \|\Gamma\|_\infty < 1$. Thus, in this case, the matrix relations (21) are fulfilled with diagonal matrix H defined by (24). This completes the proof of Theorem 5.1. ■

Note that in the proof of Theorem 5.1, we in essence use the fact of existence of a majorant linear time-invariant system

$$x(k+1) = \hat{A}x(k), \qquad k = 0, 1, 2, \ldots$$

and the Lyapunov function [21], [32]

$$V_h(x) = \max_{1\le i\le n} \{h_i |x_i|\}, \qquad h_i > 0, \, i = 1, \ldots, n$$

which is a weighted infinity vector norm with weights h_i, $i = 1, \ldots, n$. In this case, we can let $\theta = \|\Gamma\|_\infty = \|H\hat{A}H^{-1}\|_\infty$ in the inequality (18).

Theorem 5.1 is a natural generalization of [5, Th. 1] and [32, Th. 2.1] for the problem of robust stability of linear time-varying interval discrete systems. The condition of Theorem 5.1 is only sufficient for robust absolute stability of (1). However, if this theorem is applied to (1) satisfying some additional conditions, Theorem 5.1 becomes a necessary and sufficient condition for special cases of (1).

We recall that from [39], a matrix $A \in R^{n\times n}$ is called a Morishima matrix if by symmetric row and column permutations it can be transformed into the form

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where $A_{11} \ge 0$, $A_{22} \ge 0$ are square submatrices and $A_{12} \le 0$, $A_{21} \le 0$. A matrix $A \in R^{n\times n}$ is a Morishima matrix if and only if $SAS = |A|$ for some matrix $S = \operatorname{diag}\{s_1, \ldots, s_n\}$ with $s_i = \pm 1$, $i = 1, \ldots, n$. If A is a Morishima matrix, then it is Schur stable if and only if $|A|$ is Schur stable.

Our next result characterizes several classes of (1) whose robust absolute stability is equivalent to Schur stability of a single test matrix $\hat{A}$ in (23).

Theorem 5.2: The sufficient condition in Theorem 5.1 is also necessary for each of the following cases of (1).

- Case 1) In the given set of vertex matrices $\tilde{A}_{i\nu}$, $i = 1, \ldots, q$; $\nu = 1, \ldots, s$; $s = 2^r$, defined by (9), there exists at least one matrix $\tilde{A}_{\hat{i}\hat{\nu}}$ such that $\tilde{A}_{\hat{i}\hat{\nu}} = \hat{A}$ or $\tilde{A}_{\hat{i}\hat{\nu}} = -\hat{A}$.
- Case 2) In the given set of vertex matrices $\tilde{A}_{i\nu}$, $i = 1, \ldots, q$; $\nu = 1, \ldots, s$; $s = 2^r$, defined by (9), there exists at least one matrix $\tilde{A}_{\hat{i}\hat{\nu}}$ such that $|\tilde{A}_{\hat{i}\hat{\nu}}| = \hat{A}$ and $\hat{A}$ is a Morishima matrix.
- Case 3) All matrices $\tilde{A}_{i\nu}$, $i = 1, \ldots, q$; $\nu = 1, \ldots, s$; $s = 2^r$, defined by (9), are either all upper triangular or all lower triangular.

Proof: In Case 1), the proof follows immediately from Corollary 4.2 and the obvious relation $\rho(-\hat{A}) = \rho(\hat{A}) = \rho(\tilde{A}_{\hat{i}\hat{\nu}}) < 1$.

In Case 2), from Corollary 4.2, it follows that the matrix $\tilde{A}_{\hat{i}\hat{\nu}}$ is Schur stable. Since $\tilde{A}_{\hat{i}\hat{\nu}}$ is a Morishima matrix, then the matrix $|\tilde{A}_{\hat{i}\hat{\nu}}| = \hat{A}$ is also Schur stable.

In Case 3), if all matrices $\tilde{A}_{i\nu}$, $i = 1, \ldots, q$; $\nu = 1, \ldots, s$; $s = 2^r$, are upper triangular, then from the definition in (23) for the matrix $\hat{A}$, it follows that it is also upper triangular and

$$\rho(\hat{A}) = \max_{1\le i\le q,\, 1\le \nu\le s} \left\{\rho\left(\tilde{A}_{i\nu}\right)\right\}.$$

From Corollary 4.2, we have $\rho(\tilde{A}_{i\nu}) < 1$, $i = 1\ldots, q$; $\nu = 1, \ldots, s$; $s = 2^r$, and consequently, $\rho(\hat{A}) < 1$, i.e., the matrix $\hat{A}$ is Schur stable.

In the case of lower triangular matrices $\tilde{A}_{i\nu}$, $i = 1\ldots, q$; $\nu = 1, \ldots, s$; $s = 2^r$, the proof is fully analogous. This completes the proof of Theorem 5.2. ■

Note that in the case of upper or lower triangular matrices $\tilde{A}_{i\nu}$, $i = 1\ldots, q$; $\nu = 1, \ldots, s$; $s = 2^r$, the robust absolute stability of (1) depends only on their diagonal elements. Therefore, the off-diagonal elements have no effect on robust absolute stability of (1) in this case. The results of Theorem 5.2 are a generalization of the results derived in [5], [32], and [39] for linear time-varying interval discrete systems. Note also that the statement of Case 3) of Theorem 5.2 is a natural consequence of the discrete-time counterpart of [25, Th. 2].

VI. Example

In this section, the use of the present results to analyze the robust absolute stability of (1) will be illustrated by a particular example.

Let us consider the case when the convex polytope $\mathcal{A}$ defined by (2) is a family of interval matrices defined by

$$\mathcal{A}(\underline{A}, \overline{A}) = \{A \in R^{n\times n}\colon \underline{A} \le A \le \overline{A}\}$$

where $\underline{A} = [\underline{a}_{ij}] \in R^{n\times n}$ and $\overline{A} = [\overline{a}_{ij}] \in R^{n\times n}$ are fixed matrices, and the inequalities are element-wise. The family $\mathcal{A}(\underline{A}, \overline{A})$ is a hyper-rectangle in the space $R^{n\times n}$ of the coefficients a_{ij}.

When $b_j c_j^T \ge 0$, $j = 1, \ldots, r$, the set $\tilde{\mathcal{A}}$ defined by (10) is contained in the family of interval matrices given by

$$\tilde{\mathcal{A}}(\underline{A}, \overline{A}) = \left\{ A \in R^{n\times n}\colon \underline{A} + \sum_{j=1}^{r} \alpha_j b_j c_j^T \le A \le \overline{A} + \sum_{j=1}^{r} \beta_j b_j c_j^T \right\}$$

and a corresponding matrix $\hat{A}$ defined by (23) for the set $\tilde{\mathcal{A}}$ is determined as

$$\hat{A} = \max\left\{ \left|\underline{A} + \sum_{j=1}^{r} \alpha_j b_j c_j^T\right|;\ \left|\overline{A} + \sum_{j=1}^{r} \beta_j b_j c_j^T\right| \right\}$$

where the maximum is element-wise. If

$$\underline{A} + \sum_{j=1}^{r} \alpha_j b_j c_j^T \ge 0$$

or

$$\left|\underline{A}+\sum_{j=1}^{r}\alpha_j b_j c_j^T\right| \le \overline{A}+\sum_{j=1}^{r}\beta_j b_j c_j^T$$

then

$$\hat{A}=\overline{A}+\sum_{j=1}^{r}\beta_j b_j c_j^T.$$

If

$$\overline{A}+\sum_{j=1}^{r}\beta_j b_j c_j^T \le 0$$

or

$$\underline{A}+\sum_{j=1}^{r}\alpha_j b_j c_j^T \le -\left|\overline{A}+\sum_{j=1}^{r}\beta_j b_j c_j^T\right|$$

then

$$\hat{A}=-\left(\underline{A}+\sum_{j=1}^{r}\alpha_j b_j c_j^T\right).$$

Therefore, the conditions of Theorem 5.2 Case 1) are fulfilled in these cases, and the necessary and sufficient condition for robust absolute stability of (1) is the inequality $\rho(\overline{A}+\sum_{j=1}^{r}\beta_j b_j c_j^T) < 1$ or $\rho(\underline{A}+\sum_{j=1}^{r}\alpha_j b_j c_j^T) < 1$, respectively.

We can obtain similar results for the case when $b_j c_j^T \le 0$, $j=1,\ldots,r$. In this case, we only need to replace α_j in the above formulas by β_j, $j=1,\ldots,r$, and vice versa.

VII. Conclusions

In this paper, we have studied the problem of robust absolute stability of a class of nonlinear discrete-time systems with time-varying matrix uncertainty of polyhedral type and with multiple time-varying sector nonlinearities. By using the variational method and the discrete version of Lyapunov Second Method, necessary and sufficient conditions for robust absolute stability have been obtained in different forms for the given class of systems. It was shown that in general, the problem of checking these conditions can be effectively solved by numerical methods, admitting a computer-aided implementation. Several simple sufficient conditions for robust absolute stability have been provided which become necessary and sufficient for several special cases. As an example, we have applied the present results to a specific class of systems with time-varying interval matrices in the linear part.

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