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STABILITY ANALYSIS OF A CLASS OF SYSTEMS WITH PARAMETER UNCERTAINTIES AND WITH STATE SATURATION NONLINEARITIES

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SUMMARY

For linear systems with parameter uncertainties and subject to state saturation, we establish results concerning the global asymptotic stability of an equilibrium. In addition to providing a means for a qualitative analysis, these results also enable us to address the stabilizability of such systems by means of linear state feedback.

Systems of the type considered herein capture two important phenomena commonly encountered in the modelling process: (i) system parameter uncertainties (which in the present case are modelled by means of interval matrices), and (ii) operation of systems over a wide range (which in the present case is accounted for by state saturation nonlinearities).

We demonstrate the applicability of the present results by means of several specific examples.

KEY WORDS saturation; interval matrix; parameter uncertainties; stability analysis

1. INTRODUCTION

In this paper, we first investigate the stability properties of a class of systems described by

$$x(k+1) = \text{sat}[Ax(k)], \qquad k = 0, 1, 2, \ldots \tag{1}$$

where $x(k) \in R^n$, $A = [a_{ij}] \in R^{n \times n}$, $\text{sat}(x) = [\text{sat}(x_1), \text{sat}(x_2), \ldots, \text{sat}(x_n)]^T$, and

$$\text{sat}(x_i) = \begin{cases} 1, & x_i > 1 \\ x_i, & -1 \leq x_i \leq 1 \\ -1, & x_i < -1 \end{cases}$$

We assume that in system (1), the matrix A is known to belong to an interval matrix, i.e. $A \in [A^m, A^M]$. (An *interval matrix* $[A^m, A^M]$ with $A^m = [a_{ij}^m] \in R^{n\times n}$, $A^M = [a_{ij}^M] \in R^{n\times n}$, and $a_{ij}^m \leq a_{ij}^M$ for all i and j is defined by $[A^m, A^M] \triangleq \{C = [c_{ij}] \in R^{n\times n}: a_{ij}^m \leq c_{ij} \leq a_{ij}^M, 1 \leq i, j \leq n\}$.) We will refer to system (1) as a 'system with saturation nonlinearities and parameter uncertainties'. Because of the presence of saturation nonlinearities in system (1), it is clear that for any $x(0) \notin D^n \triangleq \{x \in R^n: -1 \leq x_i \leq 1,\ i = 1, \ldots, n\}$, $x(k) \in D^n$ for $k \geq 1$ will always be true. Thus, without loss of generality, we will assume that $x(0) \in D^n$.

By using the stability results for system (1), we will establish sufficient conditions for the *linear feedback stabilization* of control systems described by

$$x(k+1) = \text{sat}[Ax(k) + Bu(k)], \qquad k = 0, 1, 2, \ldots, \tag{2}$$

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where $u(k) \in D^m$, $B = [b_{ij}] \in R^{n \times m}$, and $x(k)$, A, and the function sat$(\cdot)$ are defined in equation (1). We refer to such systems as 'systems with control constraints, state saturation and parameter uncertainties'.

Systems described by equation (2) arise frequently in the modelling of certain control systems. Systems described by equation (1) can be considered as such control systems with no external inputs. Examples of systems (1) and (2) include mechanical systems with speed and position limits, electric motor systems with limited power supply, and many other process control systems. These types of systems capture three phenomena which are encountered in systems descriptions: (i) system parameter uncertainties, which in the present case are modelled by interval matrices; (ii) operation of systems over a wide range, which in the present case is accounted for by saturation nonlinearities; and (iii) the use of constrained controls in consideration of the limited energy in control signals. Generally speaking, state saturation nonlinearities in control systems are very common in practice (see, for example Reference 5). Parameter uncertainties also arise very often in system modelling due to, among other things, modelling errors and measurement errors. In the analysis and design of systems described by (1), the most basic question addresses stability: under what conditions is $x_e = 0$ an equilibrium and when is this equilibrium globally asymptotically stable? On the other hand, in the analysis and design of systems described by (2) (in which the control signal is constrained in a hypercube), system stabilizability is of fundamental importance.

Systems with saturation nonlinearities (and with no parameter uncertainties) have been investigated by many researchers (see, for example, References 2, 5–7, 11, 15, 16). Systems with parameter uncertainties characterized by interval matrices have also been widely investigated (see, for example, References 1, 3, 4, 8, 12, 14, 17–21). The stability and stabilizability of systems with saturation nonlinearities *and* parameter uncertainties do not appear to have been addressed. We intend to investigate such problems in the present paper. Among the above citations, References 3, 4, 7, 14, 17 and 20 include results which are perhaps most closely related to the present work, making use of quadratic Lyapunov functions to establish sufficient conditions (involving the testing of a finite number of matrices) to ensure the stability of an equilibrium of linear systems with parameter uncertainties (characterized by interval matrices). In particular, as in the present case, discrete-time systems are considered in References 7 and 20.

This paper is organized as follows. In Section 2, we introduce some necessary notation. In Section 3, we first establish our main results for the stability analysis of system (1) (refer to Lemmas 1 and 2 and Theorems 1, 2, and 3). These results are then applied (also in Section 3) to establish conditions for the stabilizability of system (2), using linear state feedback (refer to Lemma 3 and Corollary 1). In Section 4, we demonstrate the applicability of the present results by means of several specific examples (refer to Examples 1, 2, and 3). In Section 5, we conclude the present paper with a few pertinent remarks.

2. NOTATION

Before presenting our results, we introduce some necessary notation.

Let V and W be arbitrary sets. Then $V \cup W$, $V \cap W$, $V \setminus W$, and $V \times W$ denote the union, intersection, difference, and Cartesian product of V and W, respectively. If V is a subset of W, we write $V \subset W$ and if x is an element of V, we write $x \in V$. If $x, y \in R^n$, then $x \leq y$ signifies that $x_i \leq y_i$, $x < y$ signifies that $x_i < y_i$, and $x > 0$ signifies that $x_i > 0$ for all $i = 1, \ldots, n$.

For $x \in R^n$, we define the l_p vector norm as

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}, \quad \text{for } 1 \leq p \leq \infty \tag{3}$$

For $A \in R^{n \times n}$ and for an arbitrary vector norm $\|\cdot\|$, we define the norm of A by

$$\|A\| = \inf\{\gamma : \|Ax\| \leq \gamma \|x\| \text{ for all } x \in R^n\}$$

In particular, for $p = 1, 2$ and ∞, the norms of A, induced by the l_p vector norm, are given by

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}|$$

$$\|A\|_2 = \sqrt{\lambda_M(A^{\mathrm{T}}A)}$$

and

$$\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|$$

where $\lambda_M(A^{\mathrm{T}}A)$ denotes the largest eigenvalue of $A^{\mathrm{T}}A$.

A square matrix is said to be *positive definite*, if it is symmetric and if all its eigenvalues are positive. A matrix $H = [h_{ij}] \in R^{n \times n}$ is said to be *diagonally dominant*, if

$$|h_{ii}| \geq \sum_{j=1, j \neq i}^{n} |h_{ij}| \quad \text{for } i = 1, \ldots, n \tag{4}$$

where $|\cdot|$ denotes absolute value.

We let I and J denote subsets of $\{1, \ldots, n\}$ satisfying the following properties (*i*) $I \cup J = \{1, \ldots, n\}$ and (*ii*) $I \cap J = \phi$, where I and J are allowed to be ϕ, but not simultaneously. With I, J specified as above, we denote

$$R_{IJ} = \{x \in R^n : x_I \geq 0 \quad \text{and} \quad x_J < 0\} \tag{5}$$

where $I = \{i_1, \ldots, i_s\}$, $J = \{j_1, \ldots, j_{n-s}\}$, $x_I = [x_{i_1}, \ldots, x_{i_s}]^{\mathrm{T}}$, and $x_J = [x_{j_1}, \ldots, x_{j_{n-s}}]^{\mathrm{T}}$. For the set $\{1, \ldots, n\}$, there are 2^n *different* pairs (I, J). Note that

$$R^n = \bigcup_{(I,J)} R_{IJ}$$

For I, J, I' and J' such that

$$\begin{aligned} I \cup J = I' \cup J' &= \{1, \ldots, n\} \\ I \cap J = I' \cap J' &= \phi \end{aligned} \tag{6}$$

we let $A^{IJ}_{I'J'} = [a_{ij}^{IJI'J'}] \in R^{n \times n}$, where

$$a_{ij}^{IJI'J'} = \begin{cases} a_{ij}^M & \text{if } (i,j) \in I' \times I \cup J' \times J \\ a_{ij}^m & \text{if } (i,j) \in I' \times J \cup J' \times I \end{cases} \tag{7}$$

with the convention that $I' \times I = \phi$ when $I' = \phi$ or when $I = \phi$.

Let $v : B \to R$ be continuous where $B \subset R^n$ is a convex set. We call the function $v(x)$ a *convex function* if for any finite set of points $x_1, \ldots, x_l$ in B and any $0 \leq \lambda_1, \ldots, \lambda_l \leq 1$ with $\sum_{i=1}^{l} \lambda_i = 1$, we have

$$v\left(\sum_{i=1}^{l} \lambda_i x_i\right) \leq \sum_{i=1}^{l} \lambda_i v(x_i)$$

3. MAIN RESULTS

We first establish some preliminary results.

Consider the system described by

$$x(k+1) = f(x(k)), \qquad k = 0, 1, \ldots \tag{8}$$

where $x(k) \in R^n$, $f: R^n \to R^n$ and f is assumed to satisfy

$$f_m(x(k)) \leq f(x(k)) \leq f_M(x(k)) \tag{9}$$

where f_m and f_M are given functions and (9) is interpreted componentwise.

A point $x_e \in R^n$ is called an *equilibrium* of the system determined by equation (8) if $x_e = f(x_e)$, where f is known to satisfy (9). We can assume, without loss of generality, that $x_e = 0$ (see, for example, Reference 13).

Definition 1

The equilibrium $x_e = 0$ of system (8) and is said to be *globally asymptotically stable* if (i) it is *stable*, i.e. for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon)$ such that $\|x(k)\| < \varepsilon$ for all $k = 0, 1, 2, \ldots$, whenever $\|x(0)\| < \delta$ ($\|\cdot\|$ denotes any vector norm), and (ii) it is *attractive*, i.e., $x(k) \to 0$ as $k \to \infty$.

We recall that the equilibrium $x_e = 0$ for system (8) is globally asymptotically stable, if there exists a continuous function $v: R^n \to R$ which is positive definite, radially unbounded, and along solutions of (8),

$$Dv_{(8)}(x(k)) \triangleq v(x(k+1)) - v(x(k)) = v(f(x(k))) - v(x(k)) \tag{10}$$

is negative-definite for all $x(k) \in R^n$. Such a function v is an example of a *Lyapunov function.*

Definition 2

A system $x(k+1) = f_e(x(k))$ is said to be an *extreme system* of the system determined by (8) and (9) if each component of f_e is a corresponding component of either f_m or f_M (i.e. for $f_e = (f_e^1, \ldots, f_e^n)^T$, $f_m = (f_m^1, \ldots, f_m^n)^T$, $f_M = (f_M^1, \ldots, f_M^n)^T$ and $I = \{i_1, \ldots, i_s\}$, $J = \{j_1, \ldots, j_{n-s}\}$, such that $I \cup J = \{1, \ldots, n\}$, and $I \cap J = \phi$, we have $f_e^i = f_m^i$ for $i \in I$ and $f_e^j = f_M^j$ for $j \in J$ with $I = \phi$ or $J = \phi$ allowed).

It is clear that there are 2^n extreme systems for the system determined by (8) and (9). We denote these 2^n extreme systems by

$$x(k+1) = f_{e_i}(x(k)) \tag{e_i}$$

$i = 1, \ldots, 2^n$.

The following lemma is required in establishing our first result.

Lemma 1

Suppose that $v: R^n \rightarrow R$ is a convex function. Then,

$$Dv_{(8)}(x(k)) = v(f(x(k))) - v(x(k)) \leq \max_{1 \leq i \leq 2^n} \{Dv_{(e_i)}(x(k))\} \tag{11}$$

for any $x(k) \in R^n$, where $Dv_{(e_i)}(x(k)) \triangleq v(f_{e_i}(x(k))) - v(x(k))$.

Proof. From (9), we see that for any $x(k) \in R^n$, $f(x(k))$ is in the (closed) hypercuboid in R^n with vertices $f_{e_i}(x(k))$, $i = 1, \ldots, 2^n$. Since a hypercuboid is a convex set, there exist $\alpha_i = \alpha_i(x(k))$, $0 \leq \alpha_i \leq 1$, $i = 1, \ldots, 2^n$, such that $\sum_{i=1}^{2^n} \alpha_i = 1$ and

$$f(x(k)) = \sum_{i=1}^{2^n} \alpha_i f_{e_i}(x(k))$$

We now have, using the convexity of function v,

$$\begin{aligned} Dv_{(8)}(x(k)) = v(f(x(k))) - v(x(k)) &= v\left(\sum_{i=1}^{2^n} \alpha_i f_{e_i}(x(k))\right) - v(x(k)) \\ &\leq \sum_{i=1}^{2^n} \alpha_i v(f_{e_i}(x(k))) - \sum_{i=1}^{2^n} \alpha_i v(x(k)) = \sum_{i=1}^{2^n} \alpha_i Dv_{(e_i)}(x(k)) \\ &\leq \max_{1 \leq i \leq 2^n} \{Dv_{(e_i)}(x(k))\} \end{aligned}$$

for any $x(k) \in R^n$. □

Associated with the nonlinear system (1), we will consider linear interval systems given by

$$w(k+1) = Aw(k) \tag{12}$$

where $w \in R^n$ and A is defined in (1) (i.e., $A \in [A^m, A^M]$). For $I = \{i_1, \ldots, i_s\}$ and $J = \{j_1, \ldots, j_{n-s}\}$ satisfying $I \cup J = \{1, \ldots, n\}$ and $I \cap J = \phi$ (refer to (5) for an interpretation of the index sets I, J), we define

$$A^{IJ}_L = [\underline{a}^{IJ}_{kl}] \in R^{n \times n}$$

$$\underline{a}^{IJ}_{kl} \triangleq \begin{cases} a^m_{kl} & \text{for } l \in I \\ a^M_{kl} & \text{for } l \in J \end{cases}$$

and

$$A^{IJ}_U = [\bar{a}^{IJ}_{kl}] \in R^{n \times n}$$

$$\bar{a}^{IJ}_{kl} \triangleq \begin{cases} a^M_{kl} & \text{for } l \in I \\ a^m_{kl} & \text{for } l \in J \end{cases}$$

Since

$$R^n = \bigcup_{(I,J)} R_{IJ}$$

it can be seen that, for $w \in R^n$, $A \in [A^m, A^M]$ implies that

$$A_L^{IJ} w \leq Aw \leq A_U^{IJ} w, \qquad w \in R_{IJ}$$

where A_L^{IJ} and A_U^{IJ} are defined by (13) and (14), respectively.

For $w \in R^n$, $I' \cup J' = \{1, \ldots, n\}$ and $I' \cap J' = \phi$, we denote

$$f_{e_{I'J'}}(w) = [f^1_{e_{I'J'}}(w), \ldots, f^n_{e_{I'J'}}(w)]^{\mathrm{T}} \tag{16}$$

where

$$f^j_{e_{I'J'}}(w) = \begin{cases} A_U^{IJ} w & \text{for } j \in I' \text{ and } w \in R_{IJ} \\ A_L^{IJ} w & \text{for } j \in J' \text{ and } w \in R_{IJ} \end{cases}$$

A straightforward calculation yields

$$f_{e_{I'J'}}(w) = A^{IJ}_{I'J'} w, \qquad w \in R_{IJ} \tag{18}$$

where $A^{IJ}_{I'J'}$ is defined in (7). From (16) and (17), we see that each component of $f_{e_{I'J'}}(w)$ is either the lower bound or the upper bound in (15). By Definition 2,

$$w(k+1) = f_{e_{I'J'}}(w(k)) = A^{IJ}_{I'J'} w(k), \; w(k) \in R_{IJ} \tag{$e_{I'J'}$}$$

represents the 2^n extreme systems for the system given by (12). Therefore, equation (11) in Lemma 1, when applied to system (12), becomes

$$Dv_{(12)}(w(k)) \leq \max_{(I',J')} \{Dv_{(e_{I'J'})}(w(k))\}, \; w(k) \in R_{IJ} \tag{19}$$

The following lemma is also required in establishing our results.

Lemma 2

Let $x_s = \mathrm{sat}(x) = [\mathrm{sat}(x_1), \ldots, \mathrm{sat}(x_n)]^{\mathrm{T}}$ for $x \in R^n$ and let $H = [h_{ij}] \in R^{n \times n}$ denote a positive-definite matrix. Then, $x_s^{\mathrm{T}} H x_s < x^{\mathrm{T}} H x$ is true for all $x \in R^n \setminus D^n$ if and only if H satisfies the diagonal dominance condition (4), where D^n is the closed hypercube in R^n.

Proof. See Reference 6 □

We will utilize the following definition assumption in our results.

Definition 3

An interval matrix $[A^m, A^M] \subset R^{n \times n}$ is said to possess *Property* $\mathfrak{R}$ if there exists a positive-definite matrix $H \in R^{n \times n}$ which satisfies the diagonal dominance condition (4) such that

$$Q = H - (A^{IJ}_{I'J'})^{\mathrm{T}} H A^{IJ}_{I'J'} \tag{20}$$

is positive-definite for all I, J, I' and J' satisfying (6), where $A^{IJ}_{I'J'}$ is defined in (7).

Remark 1

A matrix $W = [w_{ij}] \in [A^m, A^M]$ with $w_{ij} = a^m_{ij}$ or $w_{ij} = a^M_{ij}$ for all $i, j = 1, \ldots, n$ is called an *extreme matrix* of the interval matrix $[A^m, A^M]$. It is clear that there are 2^{n^2} extreme matrices in

$[A^m, A^M]$. It can easily be shown that $A^{IJ}_{I'J'}$ is an extreme matrix of $[A^m, A^M]$ and that there are 2^{2n} different $A^{IJ}_{I'J'}$ where I, J, and I', J' satisfy (6). From (7) we see that $A^{IJ}_{I'J'} = A^{JI}_{J'I'}$. Thus, there are only 2^{2n-1} distinct extreme matrices $A^{IJ}_{I'J'}$. Therefore, to verify the Property $\mathfrak{R}$ in Definition 3, we need to check the positive-definiteness of 2^{2n-1} different matrices.

Remark 2

We note that Lemma 1 implies that the equilibrium $x_e = 0$ of linear system (12) is asymptotically stable if Q in (20) is positive-definite for 2^{2n-1} distinct extreme matrices $A^{IJ}_{I'J'}$.

We are now in a position to establish the next theorem.

Theorem 1

The equilibrium $x_e = 0$ of system (1) is *globally asymptotically stable* if the interval matrix $[A^m, A^M]$ possesses Property $\mathfrak{R}$.

Proof. Choose $v(w) = w^{\mathrm{T}}Hw$ for system (12). Since $\sqrt{v(w)}$ is a norm on R^n, and since any norm is a convex function, it follows that $\sqrt{v(w)}$ is convex. Furthermore, it can easily be proved, using the definition of convex function, that the square of any convex function is also convex. Therefore, $v(w)$ is convex. By Lemma 1 and equation (19), we now compute for $w(k) \in R^n$,

$$v(Aw(k)) - v(w(k)) = Dv_{(12)}(w(k)) \leq \max_{(I',J')} \{Dv_{(e_{I'J'})}(w(k))\}$$

$$\leq \max_{(I,J,I',J')} \{w^{\mathrm{T}}(k)[(A^{IJ}_{I'J'})^{\mathrm{T}}HA^{IJ}_{I'J'} - H]w(k)\}$$

where we have used, for $w(k) \in R_{IJ}$,

$$Dv_{(e_{I'J'})}(w(k)) = v(f_{e_{I'J'}}(w(k))) - v(w(k)) = \mathrm{w}^{\mathrm{T}}(k)[(A^{IJ}_{I'J'})^{\mathrm{T}}HA^{IJ}_{I'J'} - H]w(k)$$

Now, we consider $v(x) = x^{\mathrm{T}}Hx$ for system (1). From Lemma 2, we see that

$$v(\mathrm{sat}(x)) \leq v(x) \text{ for any } x \in R^n \tag{22}$$

since H satisfies the diagonal dominance condition (4). Using (21), we compute for $x(k) \in R^n$,

$$Dv_{(1)}(x(k)) = v(\mathrm{sat}(Ax(k))) - v(x(k)) \leq v(Ax(k)) - v(x(k))$$

$$\leq \max_{(I,J,I',J')} \{x^{\mathrm{T}}(k)[(A^{IJ}_{I'J'})^{\mathrm{T}}HA^{IJ}_{I'J'} - H]x(k)\} \tag{23}$$

The hypotheses in the theorem imply that the right-hand side of (23) is negative for all $x(k) \in R^n \setminus \{0\}$. Then, $Dv_{(1)}(x(k))$ is negative-definite for all $x(k) \in R^n$ and for all $A \in [A^m, A^M]$. Therefore, by the Second Method of Lyapunov, we see that the equilibrium $x_e = 0$ of system (1) is globally asymptotically stable. □

Our next result utilizes the following definition.

Definition 4

An interval matrix $[A^m, A^M] \subset R^{n\times n}$ is said to possess *Property* $\mathfrak{T}$ if there exists a positive-definite matrix $H \in R^{n\times n}$ which satisfies the diagonal dominance condition (4) such that

$$Q = H - A_0^{\mathrm{T}}HA_0 \tag{24}$$

is positive-definite, where $A_0 \triangleq \frac{1}{2}(A^m + A^M)$, and if

$$\tfrac{1}{2}\alpha(A^M - A^m) < \sqrt{(\alpha(A_0))^2 + \frac{\lambda_m(Q)}{\|H\|_\infty}} - \alpha(A_0)$$

where $\alpha(\cdot)$ is defined as

$$\alpha(W) = \max\{\|W\|_1, \|W\|_\infty\} \text{ for } W \in R^{n \cdot n}$$

and $\lambda_m(Q)$ is the smallest eigenvalue of Q.

Theorem 2

The equilibrium $x_e = 0$ of system (1) is *globally asymptotically stable* if the interval matrix $[A^m, A^M]$ possesses Property $\mathcal{T}$.

Proof. Choose $v(x) = x^{\mathrm{T}}Hx$ for system (1). It suffices to show that $Dv_{(1)}(x(k))$ is negative-definite for all $x(k) \in R^n$ and for all $A \in [A^m, A^M]$. For $x(k) \in R^n$, we compute

$$Dv_{(1)}(x(k)) = v(\mathrm{sat}(Ax(k))) - v(x(k)) \leq v(Ax(k)) - v(x(k)) = x^{\mathrm{T}}(k)(A^{\mathrm{T}}HA - H)x(k)$$

where we have used equation (22). We will show that $A^{\mathrm{T}}HA - H$ is negative-definite for all $A \in [A^m, A^M]$.

Let $\Delta A = A - A_0 = A - \frac{1}{2}(A^m + A^M)$. Then $\Delta A = [\Delta a_{ij}] \in R^{n \times n}$ satisfies the relation

$$|\Delta a_{ij}| = |a_{ij} - \tfrac{1}{2}(a_{ij}^m + a_{ij}^M)| \leq \tfrac{1}{2}(a_{ij}^M - a_{ij}^m) \tag{27}$$

for all $1 \leq i, j \leq n$, where we have used the fact that $a_{ij}^m \leq a_{ij} \leq a_{ij}^M$. We note that $\frac{1}{2}(a_{ij}^M - a_{ij}^m)$ is the (i, j)th element of $\frac{1}{2}(A^M - A^m)$ and (27) implies that

$$\|\Delta A\|_1 \leq \tfrac{1}{2}\|A^M - A^m\|_1$$

and

$$\|\Delta A\|_\infty \leq \tfrac{1}{2}\|A^M - A^m\|_\infty$$

Thus, we have

$$\alpha(\Delta A) \leq \tfrac{1}{2}\alpha(A^M - A^m)$$

where $\alpha(\cdot)$ is defined in (26).

Next, we compute

$$\begin{aligned} A^{\mathrm{T}}HA - H &= (A_0 + \Delta A)^{\mathrm{T}}H(A_0 + \Delta A) - H \\ &= A_0^{\mathrm{T}}HA_0 + A_0^{\mathrm{T}}H(\Delta A) + (\Delta A)^{\mathrm{T}}HA_0 + (\Delta A)^{\mathrm{T}}H(\Delta A) - H \\ &= -Q + A_0^{\mathrm{T}}H(\Delta A) + (\Delta A)^{\mathrm{T}}HA_0 + (\Delta A)^{\mathrm{T}}H(\Delta A) \end{aligned}$$

The matrix $A^{\mathrm{T}}HA - H$ is negative-definite if

$$\lambda_M(A_0^{\mathrm{T}}H(\Delta A) + (\Delta A)^{\mathrm{T}}HA_0 + (\Delta A)^{\mathrm{T}}H(\Delta A)) < \lambda_m(Q)$$

where $\lambda_M(\cdot)$ denotes the largest eigenvalue. It suffices to show that

$$\|A_0^{\mathrm{T}}H(\Delta A) + (\Delta A)^{\mathrm{T}}HA_0 + (\Delta A)^{\mathrm{T}}H(\Delta A)\|_\infty < \lambda_m(Q)$$

since $\lambda_M(W) \leq \|W\|_\infty$ for any $W = W^{\mathrm{T}} \in R^{n \times n}$ (cf. Reference 9). We note that

$$\begin{aligned} &\|A_0^{\mathrm{T}}H(\Delta A) + (\Delta A)^{\mathrm{T}}HA_0 + (\Delta A)^{\mathrm{T}}H(\Delta A)\|_\infty \\ &\quad \leq \|A_0^{\mathrm{T}}\|_\infty\|H\|_\infty\|\Delta A\|_\infty + \|(\Delta A)^{\mathrm{T}}\|_\infty\|H\|_\infty\|A_0\|_\infty + \|(\Delta A)^{\mathrm{T}}\|_\infty\|H\|_\infty\|\Delta A\|_\infty \\ &\quad = (\|A_0\|_1\|\Delta A\|_\infty + \|\Delta A\|_1\|A_0\|_\infty + \|\Delta A\|_1\|\Delta A\|_\infty)\|H\|_\infty \\ &\quad \leq (2\alpha(A_0).\alpha(\Delta A) + [\alpha(\Delta A)]^2)\|H\|_\infty \end{aligned}$$

Let $s = \alpha(\Delta A) \geqslant 0$. By (30), it suffices to show that

$$s^2 + 2\alpha(A_0)s < \frac{\lambda_m(Q)}{\|H\|_\infty} \tag{31}$$

to prove that (29) is true. By (25) and (28), we have

$$s \leqslant \tfrac{1}{2}\alpha(A^M - A^m) < \sqrt{(\alpha(A_0))^2 + \frac{\lambda_m(Q)}{\|H\|_\infty}} - \alpha(A_0)$$

Consider equation

$$r^2 + 2\alpha(A_0)r - \frac{\lambda_m(Q)}{\|H\|_\infty} = 0$$

The two roots of the above equation are

$$r_1 = -\alpha(A_0) - \sqrt{(\alpha(A_0))^2 + \frac{\lambda_m(Q)}{\|H\|_\infty}}$$

and

$$r_2 = -\alpha(A_0) + \sqrt{(\alpha(A_0))^2 + \frac{\lambda_m(Q)}{\|H\|_\infty}}$$

Clearly $r_1 < 0 \leqslant s < r_2$, which implies that

$$s^2 + 2\alpha(A_0)s - \frac{\lambda_m(Q)}{\|H\|_\infty} < 0$$

i.e., (31) is true. This completes the proof of the theorem. □

Remark 3

Theorem 1 is in general less restrictive than Theorem 2; however, Theorem 2 is considerably easier to apply than Theorem 1, since Theorem 2 involves less computational complexity. If in system (1), $A^m = A^M = \bar{A}$, i.e., the parameters of the system are known exactly, then, Theorem 1 and Theorem 2 reduce to the same result, in which case we have $A^{IJ}_{I'J'} = A_0 = \bar{A}$.

Remark 4

In applications of Theorems 1 and 2, a question which arises naturally is how to determine a matrix H (if it exists) which satisfies the diagonal dominance condition (4) for a given interval matrix $[A^m, A^M]$. One way of solving this problem is to use linear programming. We will demonstrate this idea in Section 4 by means of a specific example.

We close the present section by establishing the following result.

Theorem 3

The equilibrium $x_e = 0$ of system (1) is *globally asymptotically stable*, if

(i) $\|W(A)\|_1 < 1$ or $\|W(A)\|_\infty < 1$, where $W(A) = [w_{ij}] \in R^{n \times n}$ and

$$w_{ij} = \begin{cases} a_{ij}^m & \text{if } |a_{ij}^m| > |a_{ij}^M| \\ a_{ij}^M & \text{if } |a_{ij}^m| < |a_{ij}^M| \\ a_{ij}^m \text{ or } a_{ij}^M & \text{if } |a_{ij}^m| = |a_{ij}^M| \end{cases}$$

or

(ii) $\|V(A)\|_2 < 1$, where $V(A) = [v_{ij}] \in R^{n \times n}$ and $v_{ij} = \max\{|a_{ij}^m|, |a_{ij}^M|\}$ for all i, j.

Proof. This is a consequence of Theorem 6 in Reference 8. We omit the details. □

By making use of the stability results presented in the previous section, we establish in this section some results concerning the stabilizability of systems described by (2).

Definition 5

System (2) is said to be *stabilizable* (*by means of linear state feedback*) if there exists a matrix $F \in R^{m \times n}$ such that (i) $Fx \in D^m$ for any $x \in D^n$; and (ii) for the system given by

$$x(k+1) = \text{sat}[Ax(k) + BFx(k)], \qquad k = 0, 1, 2, \ldots$$

the equilibrium $x_e = 0$ is globally asymptotically stable.

The following lemma is required in the proof of our next result. Its proof is straightforward and is therefore not included here.

Lemma 3

Suppose $F \in R^{m \times n}$. Then $Fx \in D^m$ for any $x \in D^n$ if and only if $\|F\|_\infty \leq 1$, where $\|\cdot\|_\infty$ denotes the matrix norm induced by the l_∞ vector norm.

We now establish our final result.

Corollary 1

System (2) is *stabilizable* (by means of linear state feedback) if there exists an $F \in R^{m \times n}$ with $\|F\|_\infty \leq 1$ such that

(i) the interval matrix $[A^m + BF, A^M + BF]$ possesses Property $\mathfrak{R}$; or
(ii) the interval matrix $[A^m + BF, A^M + BF]$ possesses Property $\mathcal{T}$

Proof. (i) From Lemma 3, we know that if $\|F\|_\infty \leq 1$, then $Fx \in D^m$ for any $x \in D^n$. We can substitute the linear state feedback $u(k) = Fx(k)$ into (2) to obtain

$$x(k+1) = \text{sat}[Ax(k) + BFx(k)] = \text{sat}[Gx(k)], \qquad k = 0, 1, 2, \ldots \tag{32}$$

where $G = A + BF \in [A^m + BF, A^M + BF]$. Since $[A^m + BF, A^M + BF]$ possesses Property $\mathcal{R}$, we know from Theorem 1 that for (32), $x_e = 0$ is globally asymptotically stable.

(ii) This part of the proof follows along similar lines as the proof of part (i). □

We will utilize a second-order example in the next section to demonstrate how to determine a state feedback matrix F in a given problem.

4. EXAMPLES

To demonstrate the applicability of the present results, we consider several specific examples.

Example 1

In system (1), let $A \in [A^m, A^M]$,

$$A^m = \begin{bmatrix} 0{\cdot}05 & -0{\cdot}2 & 0 & 0 \\ 0{\cdot}1 & -0{\cdot}7 & -0{\cdot}15 & 0{\cdot}46 \\ -0{\cdot}2 & 0{\cdot}1 & 0{\cdot}4 & 0 \\ -0{\cdot}05 & -0{\cdot}1 & 0{\cdot}19 & -0{\cdot}5 \end{bmatrix}$$

and

$$A^M = \begin{bmatrix} 0{\cdot}5 & 0{\cdot}03 & 0{\cdot}2 & 0{\cdot}1 \\ 0{\cdot}2 & -0{\cdot}29 & 0 & 0{\cdot}8 \\ -0{\cdot}1 & 0{\cdot}2 & 0{\cdot}8 & 0{\cdot}1 \\ 0{\cdot}1 & 0 & 0{\cdot}3 & -0{\cdot}3 \end{bmatrix}$$

Using Theorem 1, we choose

$$H = \begin{bmatrix} 1{\cdot}4 & 0 & -0{\cdot}2 & 0{\cdot}4 \\ 0 & 1{\cdot}6 & 0{\cdot}2 & -0{\cdot}4 \\ -0{\cdot}2 & 0{\cdot}2 & 1{\cdot}84 & -0{\cdot}5 \\ 0{\cdot}4 & -0{\cdot}4 & -0{\cdot}5 & 4{\cdot}1 \end{bmatrix}$$

which satisfies the diagonal dominance condition given by (4). Since $n = 4$ in the present example, there are $2^{2n-1} = 128$ matrices in equation (20) which need to be checked. We used MATLAB to generate the 128 matrices $A^{IJ}_{I'J'}$ from A^m and A^M and to verify the positive-definiteness of matrices Q in (20). In particular, the matrix given by

$$A_{124} = \begin{bmatrix} 0{\cdot}5 & -0{\cdot}2 & 0{\cdot}2 & 0 \\ 0{\cdot}1 & -0{\cdot}29 & -0{\cdot}15 & 0{\cdot}8 \\ -0{\cdot}2 & 0{\cdot}2 & 0{\cdot}4 & 0{\cdot}1 \\ 0{\cdot}1 & -0{\cdot}1 & 0{\cdot}3 & -0{\cdot}5 \end{bmatrix}$$

corresponds to the matrix

$$Q_{124} = H - A_{124}^{\mathrm{T}} H A_{124} = \begin{bmatrix} 0{\cdot}8354 & 0{\cdot}3458 & -0{\cdot}3458 & 0{\cdot}7208 \\ 0{\cdot}3458 & 1{\cdot}2892 & 0{\cdot}1966 & -0{\cdot}3698 \\ -0{\cdot}3458 & 0{\cdot}1966 & 1{\cdot}1766 & 0{\cdot}2574 \\ 0{\cdot}7208 & -0{\cdot}3698 & 0{\cdot}2574 & 1{\cdot}6306 \end{bmatrix}$$

having an eigenvalue $\lambda_m(Q) = 0{\cdot}0021 > 0$ which turns out to be the smallest eigenvalue among all the Q matrices involved. Thus, with the matrix H chosen as above, all conditions in Theorem 1 are satisfied and therefore, the equilibrium $x_e = 0$ of (1) with A^m and A^M given above is globally asymptotically stable.

The matrix H in (33) is determined by the following procedure. Compute $A_0 = \frac{1}{2}(A^m + A^M)$, let $H = [h_{ij}]$, and express $Q = [q_{ij}] = H - A_0^T H A_0$ in terms of h_{ij}. Choose $\sigma_i > 0$, $i = 1, 2, 3, 4$. Define a cost function J as

$$J = \min_{1 \le i \le 4} \left\{ \sigma_i q_{ii} - \sum_{j=1, j \ne i}^{4} \sigma_j |q_{ij}| \right\} \tag{34}$$

We can maximize the function J defined in (34) under the constraints given by equation (4) using linear programming. If the maximization of J results in $J > 0$, then, the matrix $Q = H - A_0^T H A_0$ is positive-definite. (In fact, $J > 0$ implies that Q is an M-matrix (see Reference [10], p. 47 for the definition and some properties of M-matrices). In particular then, Q is positive-definite.) In the present example, we chose $\sigma_1 = 1{\cdot}9$, $\sigma_2 = 1{\cdot}02$, $\sigma_3 = 1$, and $\sigma_4 = 1{\cdot}23$, and we determined the matrix H given in (33). (In the present example, the solution which guarantees that $J > 0$ is not unique and we picked one of them for purpose of demonstration.) We note that different choices of σ_is will result in different matrices H and for some of these, the maximization of the function J may not result in a positive-definite matrix Q. A natural choice of the σ_is is to set $\sigma_i = 1$ (which does not work for the present example). In the procedure demonstrated above, we wanted to determine a matrix H which satisfies the conditions in Definition3. To start with the matrix $A_0 = \frac{1}{2}(A^m + A^M)$ in this procedure has been found to work in most cases.

Example 2

We consider system (1) with $A \in [A^m, A^M]$ and

$$A_0 = \frac{1}{2}(A^m + A^M) = \begin{bmatrix} -1 & -0{\cdot}2 & -0{\cdot}1 & -0{\cdot}2 \\ 0{\cdot}3 & -0{\cdot}4 & -0{\cdot}1 & 0{\cdot}5 \\ 0{\cdot}1 & 0{\cdot}1 & -0{\cdot}3 & -0{\cdot}3 \\ 0{\cdot}1 & -0{\cdot}1 & 0{\cdot}2 & -0{\cdot}5 \end{bmatrix} \tag{35}$$

According to Theorem 2, we choose (using the procedure outlined in Example 1)

$$H = \begin{bmatrix} 1{\cdot}1 & 0{\cdot}4 & 0{\cdot}1 & 0{\cdot}2 \\ 0{\cdot}4 & 0{\cdot}7 & 0{\cdot}1 & 0 \\ 0{\cdot}1 & 0{\cdot}1 & 0{\cdot}8 & -0{\cdot}1 \\ 0{\cdot}2 & 0 & -0{\cdot}1 & 0{\cdot}6 \end{bmatrix}$$

which satisfies condition (4). We then compute $\alpha(A_0) = 1{\cdot}5$, $\|H\|_\infty = 1{\cdot}8$,

$$Q = H - A_0^T H A_0 = \begin{bmatrix} 0{\cdot}219 & 0{\cdot}123 & 0{\cdot}017 & 0{\cdot}025 \\ 0{\cdot}123 & 0{\cdot}468 & 0{\cdot}057 & 0{\cdot}051 \\ 0{\cdot}017 & 0{\cdot}057 & 0{\cdot}662 & -0{\cdot}077 \\ 0{\cdot}025 & 0{\cdot}051 & -0{\cdot}077 & 0{\cdot}247 \end{bmatrix}$$

$\lambda_m(Q) = 0{\cdot}1683$, and

$$\sqrt{(\alpha(A_0))^2 + \frac{\lambda_m(Q)}{\|H\|_\infty}} - \alpha(A_0) = 0{\cdot}0308$$

We conclude from Theorem 2 that

$$\tfrac{1}{2}\alpha(A^M - A^m) < 0{\cdot}0308$$

is a sufficient condition for the equilibrium $x_e = 0$ of (1) to be globally asymptotically stable if A_0 is specified as in (35).

Example 3

To apply Corollary 1 (i), we consider system (2) with $[A^m, A^M]$,

$$A^m = \begin{bmatrix} -0{\cdot}5 & 0{\cdot}4 \\ -0{\cdot}7 & 1{\cdot}1 \end{bmatrix}$$

$$A^M = \begin{bmatrix} -0{\cdot}1 & 1{\cdot}1 \\ -0{\cdot}4 & 1{\cdot}8 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

In the present case, the constraints for matrix $F = [f_1, f_2] \in R^{1\times 2}$ can be written as

$$\| F \|_\infty = | f_1 | + | f_2 | \leqslant 1$$

We choose the point $(f_1, f_2) = (0{\cdot}2,\ -0{\cdot}8)$ on the edge of the region in the $f_1 - f_2$ plane determined by the above condition. Selecting $F = [0{\cdot}2, -0{\cdot}8]$, we have $\| F \|_\infty = 1$,

$$A^m + BF = \begin{bmatrix} -0{\cdot}5 & 0{\cdot}4 \\ -0{\cdot}5 & 0{\cdot}3 \end{bmatrix}$$

and

$$A^M + BF = \begin{bmatrix} -0{\cdot}1 & 1{\cdot}1 \\ -0{\cdot}2 & 1 \end{bmatrix}$$

It can easily be verified that $[A^m + BF, A^M + BF]$ possesses Property $\mathfrak{R}$ for this specific choice of F. To see this, we compute $G_0 = A_0 + BF$, where $A_0 = \frac{1}{2}(A^m + A^M)$. We determine a matrix H which satisfies condition (4) in a similar manner as in Example 1 to obtain

$$H = \begin{bmatrix} 1 & -0{\cdot}7 \\ -0{\cdot}7 & 2{\cdot}1 \end{bmatrix}$$

Then, corresponding to every matrix $A^{ij}_{i'j'}$ (there are a total of 8), the matrix Q in Definition 3 (equation (20)) is positive-definite. In particular, the matrix

$$G_5 = \begin{bmatrix} -0{\cdot}5 & 0{\cdot}4 \\ -0{\cdot}2 & 1 \end{bmatrix}$$

yields

$$Q_5 = H - G_5^{\mathrm{T}} H G_5 = \begin{bmatrix} 0{\cdot}806 & -0{\cdot}486 \\ -0{\cdot}486 & 0{\cdot}4 \end{bmatrix}$$

The matrix Q_5 has the smallest eigenvalue $\lambda = 0{\cdot}0763$ among all the involved Q matrices. In accordance with Corollary 1(i), system (2) with (A, B) given above is stabilizable by linear state feedback.

5. CONCLUDING REMARKS

For linear systems with parameter uncertainties and subject to state saturation, we established results concerning two important issues. First, for the case when the input is zero, we established results for the global asymptotic stability of the equilibrium $x_e = 0$. Because of wide interest in such systems in engineering, our results have potentially many applications. Next, for linear systems with parameter uncertainties and subject to state saturation, we applied the above results to establish some results for the stabilizability of such systems by means of linear state feedback. We demonstrated the applicability of our results by means of several specific examples.

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