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ASYMPTOTIC STABILITY OF A CLASS OF LINEAR DISCRETE SYSTEMS WITH MULTIPLE INDEPENDENT VARIABLES*

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Abstract. This paper investigates the problem of asymptotic stability for a class of linear shift-invariant discrete systems with multiple independent variables. We establish the equivalence of this problem and that of robust stability for a class of ordinary linear shift-varying discrete systems with the matrix uncertainty set defined by the coefficient matrices of the original system. On the basis of this equivalence, by using the variational method and Lyapunov's second method, necessary and sufficient conditions for asymptotic stability are obtained in different forms for the class of systems considered. The parametric classes of Lyapunov functions which define the necessary and sufficient conditions of asymptotic stability are determined. We use the piecewise linear polyhedral Lyapunov functions of the infinity vector norm type to derive an algebraic criterion for asymptotic stability of the given class of discrete systems in the form of solvability conditions of a set of matrix equations. A simple sufficient condition of asymptotic stability is also obtained which becomes necessary and sufficient for several special cases of the discrete systems under consideration.
Key words: Discrete shift-invariant systems, shift-varying systems, asymptotic stability, robust stability, Lyapunov methods.

1. Introduction

The theory of discrete systems has been extensively studied for many years because of its wide applications in signal/image processing, data transmission, digital simulation of continuous physical systems described by partial differential equations, control of repetitive (multipass) processes, and recurrent neural networks, etc. (see, e.g. [13] and the references therein). In the present paper, we

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consider the following linear shift-invariant discrete systems described by the equations

$$\begin{cases} x(k_1+1,k_2,\ldots,k_q) = A_1 x(k_1,k_2,\ldots,k_q) \\ x(k_1,k_2+1,\ldots,k_q) = A_2 x(k_1,k_2,\ldots,k_q) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x(k_1,k_2,\ldots,k_q+1) = A_q x(k_1,k_2,\ldots,k_q) \end{cases}, \tag{1}$$

where $k = (k_1, \ldots, k_q) \in Z_+^q$, $Z_+^q = Z_+ \times \cdots \times Z_+$ is the set of q-tuples for a natural number q, $Z_+ = \{0, 1, \ldots\}$, $x(k) \in R^n$ is the state vector function of *multiple independent variables* k_i for $i = 1, \ldots, q$ and $A_i \in R^{n\times n}$ are real, fixed coefficient matrices for $i = 1, \ldots, q$.

We define the shift operator Δ_i for the function $x(k)$ with respect to the ith argument k_i by the relation

$$\Delta_i x(k) = \Delta_i x(k_1, \ldots, k_q) = x(k_1, \ldots, k_{i-1}, k_i + 1, k_{i+1}, \ldots, k_q).$$

Then we can rewrite the system (1) more compactly in the equivalent form

$$\Delta_i x(k) = A_i x(k), \quad i = 1, \ldots, q. \tag{2}$$

To the best of our knowledge, the system (2), and the more general nonlinear shift-varying system

$$\Delta_i x(k) = f_i[x(k), k], \quad i = 1, \ldots, q, \tag{3}$$

were considered first in [8], [9], where $f_i: R^n \times Z_+^q \to R^n$. We note that for such models, we need to have only *one* initial value for state variables to define a solution and we do not need to have boundary values. However, in spite of the only one initial value, the main characteristic feature of these models is the nonuniqueness of their solutions in the general case for given initial conditions, and, as a consequence, the existence of a set of solutions of the system under consideration starting from the same initial point. In this sense, such systems are similar to discrete systems (with a single independent variable) with parametric matrix uncertainty, and this circumstance is essentially used in the present paper.

The system (2) is different from the well-known state-space models of linear two-dimensional discrete systems including Roesser's model [27], the Fornasini-Marchesini models [6], [7], and their discrete generalizations. Moreover, the system (2) cannot be reduced to these well-known models.

The type of discrete system described by equations (2) and (3) arises when studying partial differential equations of the form

$$\partial x/\partial t_i = B_i x, \quad i = 1, \ldots, q, \tag{4}$$

and

$$\partial x/\partial t_i = g_i(x, t), \quad i = 1, \ldots, q, \tag{5}$$

respectively. In particular, solutions of (4) and (5) using digital computers will require the discretization of (4) and (5) that leads to systems of type (2) and (3), respectively. For example, using a discrete approximation based on the first-order

Euler scheme with the partial derivative $\partial x/\partial t_i$ in (4) replaced by the corresponding finite difference quotient (with a fixed $\tau > 0$), we have

$$\delta_i(x, \tau) = \tau^{-1}\left[x(t_1, \dots, t_{i-1}, t_i + \tau, t_{i+1}, \dots, t_q) - x(t_1, \dots, t_q)\right],$$

where $x(t) = x(t_1, \dots, t_q) \in R^n$ is the state vector function of the vector argument $t = (t_1, \dots, t_q)^T \in R^q$. In this case, the connection between the matrices A_i, $i = 1, \dots, q$, in (2) and the matrices B_i, $i = 1, \dots, q$, in (4) is given by the relation

$$A_i = I_n + \tau B_i, \quad i = 1, \dots, q,$$

where I_n is an $n \times n$ identity matrix. We note that partial differential equations and their solutions are of great interest in many engineering disciplines [1], [2], [17], sciences [25], [28], and other areas [24].

The notions of stability and asymptotic stability of systems (2) and (3) have been defined in [8], [9]. Using Lyapunov's second method [10], [15], sufficient conditions of stability and asymptotic stability have been established in [8], [9] for the considered systems. As for necessary and sufficient conditions, they were obtained in [8], [9] only for special classes of completely solvable linear systems (2). Complete solvability conditions for systems (2) and (3) are similar to the well-known Frobenius complete integrability conditions in the theory of partial differential equations [11]. In particular, for the shift-invariant linear system (2), the complete solvability conditions are equivalent to the pairwise commutativity of coefficient matrices of the system under study.

To the best of our knowledge, *necessary and sufficient conditions* of asymptotic stability for the general case of the linear shift-invariant discrete system (2) introduced in [8], [9], without an assumption about its complete solvability, are not available in the literature now. The main purpose of this paper is to establish such conditions in several different forms.

This paper is organized as follows. In Section 2, we introduce the notation used here and present some preliminaries. Section 3 introduces the definitions of stability, asymptotic stability, and exponential stability for the considered class of system (2). In this section, we also give a short review of results currently available in the literature and state the problem under investigation. Section 4 establishes the equivalence of the problem of *asymptotic stability* for the original class of linear shift-invariant system (2) with multiple independent variables and the problem of *robust stability* for a special class of ordinary linear shift-varying discrete systems with matrix uncertainty set defined by the matrices on the right-hand side of the original discrete system (2). This equivalence is based on the one-to-one correspondence (bijection) between the sets of solutions of both systems. In fact, the central point of this is that the discovered equivalence of the above-mentioned problems gives us opportunity to use any criteria for robust stability of the corresponding ordinary uncertain systems as criteria for asympt otic stability of the original discrete system (2). In particular, the equivalence of asymptotic and exponential stability follows immediately for the considered linear discrete system (2). The main results of this paper are presented in Section 5. Using the variational method developed in [4], [18], [19] for the problems of robust and absolute stability of discrete uncertain systems with one independent variable and Lyapunov's second method [10], [15], we establish several necessary and sufficient conditions for the asymptotic stability of the class of discrete systems under consideration. The parametric classes of Lyapunov functions which define the necessary and sufficient conditions for asymptotic stability of such systems are identified. An algebraic criterion for asymptotic stability in the form of solvability conditions of a set of matrix equations is obtained using Lyapunov functions from a class of piecewise linear polyhedral functions of the infinity vector norm type. In general, the implementation of the obtained criteria for asymptotic stability could be computationally difficult, so, in Section 6, we obtain a simple and computationally efficient sufficient condition for asymptotic stability of the discrete systems considered herein. We also show that this condition becomes necessary and sufficient for several special cases.

2. Notation and preliminaries

Let R^n denote real n-space. If $x \in R^n$, then $x^T = [x_1, \ldots, x_n]$ denotes the transpose of x. Let $R^{m\times n}$ denote the set of $m \times n$ real matrices. If $A = [a_{ij}] \in R^{m\times n}$, then A^T denotes the transpose of A. Let

$$\rho(A) = \max_{1\le i\le n} |\lambda_i(A)|$$

denote the spectral radius of the matrix $A \in R^{n\times n}$ and $\lambda_i(A)$, $i = 1, \ldots, n$, denote the eigenvalues of the matrix A.

We let $\|x\|$ denote any one of the equivalent vector norms on R^n. In particular, the l_p norms $\|x\|_p$, $1 \le p \le \infty$, are defined by

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}, \quad 1 \le p < \infty$$

and

$$\|x\|_\infty = \max_{1\le i\le n} |x_i|.$$

The matrix norm $\|A\|$ defined on $R^{m\times n}$ and induced by the vector norm $\|x\|$ on R^n, is defined as $\|A\| = \max_{\|x\|=1} \|Ax\|$. In particular, we have

$$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2 = \sqrt{\rho(A^T A)},$$

and $\|A\|_2$ is called the spectral norm of the matrix A, where

$$\|x\|_2 = \sqrt{x^T x} = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$$

is the Euclidean vector norm. We also have

$$\|A\|_\infty = \max_{\|x\|_\infty=1} \|Ax\|_\infty = \max_{1\le i\le m} \left\{ \sum_{j=1}^{n} |a_{ij}| \right\}.$$

We let $Z_+ = \{0, 1, \ldots\}$ denote the set of nonnegative integers. If q is a natural number, let $Z_+^q = Z_+ \times \cdots \times Z_+$ denote the set of q-tuples $k = (k_1, \ldots, k_q)$ with nonnegative integer components $k_i \in Z_+$, $i = 1, \ldots, q$. For two q-tuples $k^1 = (k_1^1, \ldots, k_q^1) \in Z_+^q$ and $k^2 = (k_1^2, \ldots, k_q^2) \in Z_+^q$, the notation $k^2 \ge k^1$ is understood to be component-wise inequality, i.e. $k_i^2 \ge k_i^1$ for all $i = 1, \ldots, q$. For any $k \in Z_+^q$, we denote $|k| = \sum_{i=1}^q k_i$.

Following [8], [9], we will denote by $L(k^1, \ldots, k^N)$ a set of points $k^j = (k_1^j, \ldots, k_q^j) \in Z_+^q$ satisfying the following two conditions:

1. $k^{j+1} \ge k^j,\ j = 1, \ldots, N-1;$
2. $|k^{j+1} - k^j| = |k^{j+1}| - |k^j| = \sum_{i=1}^q (k_i^{j+1} - k_i^j) = 1,\ j = 1, \ldots, N-1.$

Such a set $L(k^1, \ldots, k^N)$ will be called a *discrete curve* or a *discrete path* in Z_+^q connecting the initial point $k^1 \in Z_+^q$ and the final point $k^N \in Z_+^q$. We will also use the term *curvilinear product* of matrices $A_1, \ldots, A_q$ in (2) on the path $L(k^1, \ldots, k^N)$ for the matrix product

$$\pi_L(k^N, k^1) = \prod_{L(k^1,\ldots,k^N)} A_1^{\Delta k_1} \cdots A_q^{\Delta k_q} = \prod_{j=1}^{N-1} A_1^{k_1^{j+1}-k_1^j} \cdots A_q^{k_q^{j+1}-k_q^j}, \tag{6}$$

where $\prod_{s=1}^r D_s = D_r D_{r-1} \cdots D_1$ denotes the product of matrices $D_1, \ldots, D_r$ in the reverse order, i.e., from the right to the left. It is assumed that $\pi_L(k^1, k^1) = I_n$ if the path $L(k^1, k^1)$ consists of a single point k^1, i.e. in the case when $N = 1$.

There are many paths $L(k^1, \ldots, k^N)$ connecting the points k^1 and k^N. It is easy to check that the number of all such paths is given by

$$\frac{(|k^N| - |k^1|)!}{(k_1^N - k_1^1)! \cdots (k_q^N - k_q^1)!}.$$

In general, the matrix $\pi_L(k^N, k^1)$ in (6) depends not only on the initial and final points of the path $L(k^1, \ldots, k^N)$, but also on the intermediate points k^j, $j = 2, \ldots, N-1$. For this reason, in the general case, the system (2) does not have a unique solution $x(k, k_0, x_0)$ for a given initial point $(k_0, x_0) \in Z_+^q \times R^n$ defined for all $k \ge k_0$ such that $x(k_0, k_0, x_0) = x_0$. It is easy to verify, by the direct construction of a solution of the system (2) step by step, that the system (2) has a set of solutions $X(k, k_0, x_0) = \{x(k, k_0, x_0)\}$, and every particular solution $x(k, k_0, x_0) \in X(k, k_0, x_0)$ can be represented in the form

$$x(k, k_0, x_0) = \pi_L(k, k_0)x_0 = \prod_{L(k_0,\ldots,k)} A_1^{\Delta k_1} \cdots A_q^{\Delta k_q} x_0. \tag{7}$$

The system (2) is said to be *completely solvable* [8], [9] if, for arbitrary $k_0 \in Z_+^q$ and $x_0 \in R^n$, it has a unique solution $x(k, k_0, x_0)$ defined for all $k \geq k_0$ satisfying the condition $x(k_0, k_0, x_0) = x_0$. It is shown in [8] that the necessary and sufficient conditions of the complete solvability of the system (2) are equivalent to the pairwise commutativity of the matrices $A_1, \ldots, A_q$, i.e. to the condition

$$A_i A_j = A_j A_i \tag{8}$$

for all $i, j = 1, \ldots, q$. In this case, according to (7), a unique solution of the system (2) is given by

$$x(k, k_0, x_0) = A_1^{k_1 - k_{10}} \cdots A_q^{k_q - k_{q0}} x_0, \tag{9}$$

which easily yields the stability properties of solutions $x(k, k_0, x_0)$ of the system (2).

3. Problem statement

We now introduce definitions of stability and asymptotic stability [8], [9] for the system (2). Before their formulation, we note that if $x_0 = 0$, then the system (2) has the trivial solution $x(k, k_0, 0) \equiv 0$, $k \geq k_0$ for any discrete path $L(k_0, \ldots, k)$. Therefore, we will use the notation $x(k) \equiv 0$ for the zero solution of the system (2).

Definition 3.1. The system (2) is said to be *stable* if its zero solution $x(k) \equiv 0$ is stable, i.e. if for any $\varepsilon > 0$ and any $k_0 \in Z_+^q$ there exists a $\delta(\varepsilon, k_0) > 0$ such that the inequality $\|x_0\| < \delta(\varepsilon, k_0)$ implies that $\|x(k, k_0, x_0)\| < \varepsilon$ for any $k \geq k_0$ and for any solution $x(k, k_0, x_0) \in X(k, k_0, x_0)$ of the system (2).

In Section 4, we will show that a number $\delta(\varepsilon, k_0)$ in Definition 3.1 can be chosen independently for $k_0 \in Z_+^q$ as in the case of stable shift-invariant discrete systems.

Definition 3.2. The system (2) is said to be *asymptotically stable* if it is stable in the sense of Definition 3.1, and if the limit relation

$$\lim_{|k| \to \infty} x(k, k_0, x_0) = 0 \tag{10}$$

is fulfilled for any $x_0 \in R^n$, any $k_0 \in Z_+^q$, and any solution $x(k, k_0, x_0) \in X(k, k_0, x_0)$ of the system (2).

Definition 3.2 is formulated with the fact that for linear system (2), the concepts of local and global asymptotic stability are equivalent.

In addition to Definition 3.2, we introduce in this paper the definition of exponential stability of the system (2) as a natural generalization to the well-known definition for ordinary linear discrete systems [15].

Definition 3.3. The system (2) is said to be *exponentially stable* if there exist constants α $(0 < \alpha < 1)$ and $\beta \geq 1$, dependent only on the choice of the vector norm $\|x\|$ on R^n, such that the estimate holds

$$\|x(k, k_0, x_0)\| \leq \beta \|x_0\| \alpha^{|k|-|k_0|}, \quad k \geq k_0 \tag{11}$$

for any $k_0 \in Z_+^q$, any $x_0 \in R^n$, and any solution $x(k, k_0, x_0) \in X(k, k_0, x_0)$ of the system (2).

Comparing Definition 3.2 and Definition 3.3, it follows that the exponential stability implies asymptotic stability and, in particular, the fulfillment of the limit relation (10). As is well known [15], the asymptotic stability of ordinary linear shift-invariant discrete systems is equivalent to their exponential stability. In Section 4, we will establish the equivalence of asymptotic and exponential stability for the discrete system (2).

In [8], [9], only sufficient conditions for the asymptotic stability of discrete nonlinear system (3) and of linear system (2) have been presented based on Lyapunov's second method [10], [15] in terms of the existence of Lyapunov functions with the required properties (positive definite and strictly decreasing along the solutions of the system under consideration). For the special case of completely solvable linear system (2) satisfying the complete solvability conditions (8) in [8], necessary and sufficient conditions of asymptotic stability were derived. These conditions are reduced to the requirement of the spectral radius of every matrix A_i, $i = 1, \ldots, q$, in (2) being less than one, i.e.,

$$\rho(A_i) < 1, \quad i = 1, \ldots, q, \tag{12}$$

or all matrices A_i, $i = 1, \ldots, q$, are Schur stable. This follows directly from (9). Moreover, as is shown in [8], by direct construction of the Lyapunov function, in this case for the system (2), there exists a quadratic Lyapunov function

$$V(x) = x^T P x, \tag{13}$$

where

$$P = \sum_{j_1=0}^{\infty} \cdots \sum_{j_q=0}^{\infty} \left(A_1^T\right)^{j_1} \cdots \left(A_q^T\right)^{j_q} C A_1^{j_1} \cdots A_q^{j_q}$$

and C is an arbitrary positive definite $n \times n$ matrix. Note that a similar construction of the quadratic Lyapunov function (13) has also been obtained in [23] as a common Lyapunov function for the set of ordinary linear shift-invariant discrete systems

$$x(s+1) = A_i x(s), \quad s \in Z_+, \quad i = 1, \ldots, q \tag{14}$$

under the conditions of (8) and (12). However, necessary and sufficient conditions for asymptotic stability in the general case of the system (2) (without the assumption (8) about its complete solvability) have been unknown in the literature untill the present time. Therefore, it is the main purpose of this paper to provide such conditions in several different forms for the general case of the system (2). Our main results are presented in Sections 5 and 6. As will be shown in Section 5, from our results, the above-mentioned result for completely solvable system (2) obtained in [8] follows as a special corollary.

4. Equivalence to the problem of robust stability for ordinary linear shift-varying discrete systems

In this section, we consider along with discrete shift-invariant system (2) the ordinary linear shift-varying discrete system

$$x(s+1) = A(s)x(s), \quad s \in Z_+, \tag{15}$$

where $x(s) \in R^n$ and the shift-varying matrix $A(s) \in R^{n\times n}$ belongs for each $s \in Z_+$ to the following set of q matrices:

$$\mathcal{A} = \{A_1, \dots, A_q\} \subset R^{n\times n}, \tag{16}$$

i.e. $A(s) \in \mathcal{A}$ for all $s \in Z_+$. Note that A_i,$i = 1, 2, \dots, q$, are from the right-hand side of equation (2).

The matrix set $\mathcal{A}$ describes the structure of shift-varying parameter uncertainty in the system (15). Therefore, a solution $x(s, s_0, x_0)$ of the system (15) for an initial point $(s_0, x_0) \in Z_+ \times R^n$ is defined by an arbitrary choice of a matrix sequence

$$\{A(s) \in \mathcal{A}\colon\ s = s_0, s_0 + 1, \dots\}. \tag{17}$$

In the general case, the system (15) does not have a unique solution $x(s, s_0, x_0)$ such that $x(s_0, s_0, x_0) = x_0$, but a set of solutions defined by all possible choices of the sequence of matrices (17).

We will use the notation $x_A(s, s_0, x_0)$ to denote the solution of the system (15) starting from $x(s_0) = x_0 \in R^n$ at $s_0 \in Z_+$ that corresponds to a particular sequence of matrices (17), and the notation $X_{\mathcal{A}}(s, s_0, x_0) = \{x_A(s, s_0, x_0)\}$ to denote the set of all such solutions that correspond to all possible matrix sequences (17). In the case when $s_0 = 0$, we will use the notation $x_A(s, x_0)$ and $X_{\mathcal{A}}(s, x_0) = \{x_A(s, x_0)\}$, respectively. Note that we have $x_A(s, s_0, 0) \equiv 0$, $s = s_0, s_0 + 1, \dots,$ for any matrix sequence (17). Therefore, we will use the notation $x(s) \equiv 0$ for the zero solution of the system (15).

The equivalence of the discrete system (2) and the uncertain system (15) is established by the following lemma.

Lemma 4.1. *There is a one-to-one correspondence (bijection) between the solution sets of the systems (2) and (15).*

Proof. Note that any solution $x_A(s, s_0, x_0) \in X_{\mathcal{A}}(s, s_0, x_0)$ of the system (15) is represented in the form

$$x_A(s, s_0, x_0) = \prod_{j=s_0}^{s-1} A(j)x_0 = A(s-1)\cdots A(s_0)x_0, \quad s \geq s_0 + 1$$

or, in accordance with (16) and (17), in the form

$$x_A(s, s_0, x_0) = \prod_{\nu=1}^{s-s_0} A_{i_\nu} x_0 = A_{i_{s-s_0}} \cdots A_{i_1} x_0, \tag{18}$$

where $A_{i_\nu} \in \mathcal{A}$, $i_\nu \in \{1, \ldots, q\}$, $\nu = 1, \ldots, s - s_0$.

On the other hand, from (7) and from the definition of the discrete path $L(k_0, \ldots, k)$ it follows that any solution $x(k, k_0, x_0) \in X(k, k_0, x_0)$ of the system (2) can be represented for $|k| > |k_0|$ as

$$x(k, k_0, x_0) = \prod_{l=1}^{|k|-|k_0|} A_{i_l} x_0 = A_{i_{|k|-|k_0|}} \cdots A_{i_1} x_0, \tag{19}$$

where $A_{i_l} \in \mathcal{A}$, $i_l \in \{1, \ldots, q\}$, $l = 1, \ldots, |k| - |k_0|$.

Setting now $s_0 = |k_0|$ and $s = |k|$, from a comparison of (18) and (19), we establish the one-to-one correspondence (bijection) between the solution sets of the systems (2) and (15), which proves the lemma. □

We will say that the system (15) is *robustly stable* [4], [21] with respect to the matrix uncertainty set $\mathcal{A}$ defined by (16), if this system is globally asymptotically stable for every admissible shift-varying matrix $A(s) \in \mathcal{A}$, $s \in Z_+$. From this definition, Definition 3.2, and Lemma 4.1, we obtain the following statement.

Lemma 4.2. *The linear shift-invariant discrete system (2) is asymptotically stable if and only if the linear shift-varying system (15) is robustly stable with respect to the set $\mathcal{A}$ defined by (16).*

Thus, in accordance with Lemma 4.2, the problem of asymptotic stability of the system (2) reduces to the problem of robust stability of the linear shift-varying system (15) with respect to the set $\mathcal{A}$. Therefore, any criterion of robust stability for discrete system (15) will also be a criterion for asymptotic stability of the discrete system (2). For this reason, Lemma 4.2 is the backbone for establishing necessary and sufficient conditions of asymptotic stability of the system (2) in Section 5.

It follows from [18], [20] that the system (15) on the set $\mathcal{A}$ is equivalent to the following shift-invariant difference inclusion:

$$x(s+1) \in F(x(s)), \quad s \in Z_+, \tag{20}$$

where the multivalued vector function $F(x)$ is defined at each point $x \in R^n$ by

$$F(x) = \left\{ y \in R^n \colon \ y = Ax, \ A \in \mathcal{A} \right\}. \tag{21}$$

The equivalence is regarded in the sense of coincidence of solutions of the two systems, i.e., the coincidence of the set $X_{\mathcal{A}}(s, s_0, x_0) = \{x_A(s, s_0, x_0)\}$ of solutions of the system (15) for all admissible shift-varying matrices $A(s) \in \mathcal{A}$, $s \in Z_+$, and the set $X_F(s, s_0, x_0) = \{x_F(s, s_0, x_0)\}$ of solutions $x_F(s, s_0, x_0)$ of the difference inclusion (20), (21). Therefore, the problem of robust stability of the system (15) with respect to the set $\mathcal{A}$ and the problem of asymptotic stability of the system (2) are equivalent to the problem of global asymptotic stability of the zero solution $x(s) \equiv 0$ of the difference inclusion (20).

Because the inclusion (20) is shift invariant, the properties of stability and asymptotic stability of its zero solution $x(s) \equiv 0$ will be uniform in the initial time instant $s_0 \in Z_+$. In particular, for the stable discrete system (2) the number $\delta(\varepsilon, k_0) > 0$ in Definition 3.1 can be chosen independently for $k_0 \in Z_+^q$. Moreover, because the multivalued function $F(x)$ in (21) is homogeneous of the first order, i.e. the relation $F(\tau x) = \tau F(x)$ holds for all $x \in R^n$ and any $\tau \in R$, it follows from the results obtained in [5] that for solutions $x_F(s, s_0, x_0)$ of the globally asymptotically stable difference inclusion (20), the estimate

$$\|x_F(s, s_0, x_0)\| \le \beta \|x_0\| \alpha^{s-s_0}, \; s = s_0, s_0 + 1, \ldots \tag{22}$$

is fulfilled for any $s_0 \in Z_+$ and any $x_0 \in R^n$, and the constants α $(0 < \alpha < 1)$ and $\beta \ge 1$ in (22) depend only on the choice of vector norm $\|x\|$ on R^n.

By virtue of Lemma 4.2 and the equivalence of the system (15) to the difference inclusion (20), it follows from (22) that for solutions $x(k, k_0, x_0) \in X(k, k_0, x_0)$ of the asymptotically stable linear discrete system (2), the inequality (11) holds for any $k_0 \in Z_+^q$ and any $x_0 \in R^n$. This implies that the asymptotic stability of the system (2) is equivalent to its exponential stability, just as in the case of ordinary linear shift-invariant systems.

5. Main results

In this section, by using Lemma 4.2, we will present necessary and sufficient conditions for the asymptotic stability of the discrete system (2). Our results are in parallel to the results obtained in [4], [18]–[21] using the variational method and Lyapunov's second method.

In the framework of the variational method [4], [18], [19], we introduce for any $s \in Z_+$, the set $\Pi_s = \{\pi_s\}$ of all matrix products π_s of the form

$$\pi_s = A_{i_s} \cdots A_{i_1}, \tag{23}$$

where $\mathcal{A}_{i_\nu} \in \mathcal{A}$, $i_\nu \in \{1, \ldots, q\}$, $\nu = 1, \ldots, s$. It is assumed by definition that $\pi_0 = I_n$. Obviously, the set Π_s contains q^s matrix products π_s of the form (23).

The following theorem can now be presented.

Theorem 5.1. The system (2) is asymptotically stable if and only if there exists a finite natural number $\hat{s}$, such that

$$\|\pi_{\hat{s}}\| < 1, \quad \text{for all } \pi_{\hat{s}} \in \Pi_{\hat{s}}. \tag{24}$$

The proof of Theorem 5.1 is similar to the proof of the theorem in [4] and Lemma A.1 in [19], and is therefore omitted here.

From Theorem 5.1, the following corollary can easily be obtained.

Corollary 5.1. *The system (2) is asymptotically stable if there exists in R^n a vector norm $\|x\|$ such that the induced matrix norm $\|A_i\| < 1$ for any matrix A_i, $i = 1, \ldots, q$, in (2).*

The proof of Corollary 5.1 is obvious because

$$\Pi_1 = \mathcal{A} = \{A_1, \ldots, A_q\},$$

and under the condition $\|A_i\| < 1$, $i = 1, \ldots, q$, we will have the inequality (24) with $\hat{s} = 1$.

It is well known that the condition $\rho(A) < 1$ for a Schur stable matrix $A \in R^{n\times n}$ is equivalent to the inequality $\|A^{\hat{s}}\| < 1$ for some finite integer $\hat{s} \geq 1$. Because $A_i^{\hat{s}} \in \Pi_{\hat{s}}$ for any matrix A_i, $i = 1, \ldots, q$, in (2), it follows from (24) that $\|A_i^{\hat{s}}\| < 1$, $i = 1, \ldots, q$, and therefore, the condition (12) is satisfied. As a result, we obtain the following corollary.

Corollary 5.2. *For the asymptotic stability of the system (2), it is necessary that the condition (12) be fulfilled, i.e. that every matrix A_i, $i = 1, \ldots, q$, in (2) be Schur stable.*

Note that the necessity of the condition (12) for asymptotic stability of the system (2) also follows directly from Lemma 4.2, because in this case every linear shift-invariant system in (14) is asymptotically stable.

The next two corollaries single out special cases when the condition (12) is not only necessary but also sufficient for asymptotic stability of the system (2). Other special cases are given in Theorem 6.2 of this paper.

Corollary 5.3. *If the matrices A_i, $i = 1, \ldots, q$, in (2) are pairwise commutative, i.e. if the condition (8) is satisfied for all $i, j = 1, \ldots, q$, then for asymptotic stability of the system (2), it is necessary and sufficient that the condition (12) be fulfilled.*

Proof. Necessity follows from Corollary 5.2. For the proof of sufficiency, we note that under the condition (8), any matrix $\pi_s \in \Pi_s$ in (23) can be represented in the form

$$\pi_s = \prod_{i=1}^{q} A_i^{s_i}; \quad s_i \in Z_+; \quad i = 1, \ldots, q; \quad \sum_{i=1}^{q} s_i = s.$$

From (12) it follows that there exist constants α $(0 < \alpha < 1)$ and $\beta \geq 1$ such that

$$\|A_i^s\| \leq \beta\alpha^s, \ s \in Z_+, \ i = 1, \ldots, q.$$

Therefore, we have the inequality

$$\|\pi_s\| \leq \beta^q \alpha^s, \text{ for all } \pi_s \in \Pi_s, \quad s \in Z_+. \tag{25}$$

By choosing the integer $\hat{s} > q\ln\beta/|\ln\alpha|$, we obtain from (25) the inequality (24), which completes the proof of Corollary 5.3. □

As noted in Section 3, the sufficiency of Corollary 5.3 can also be proved by using the quadratic Lyapunov function (13) proposed in [8], [23].

We recall that a real matrix $A \in R^{n\times n}$ is said to be normal if it is commutative with its transpose, i.e. if $AA^T = A^T A$ [12], [16]. For the spectral norm of the normal matrix A, the relation $\|A\|_2 = \rho(A)$ is valid. Therefore, using this property and Corollaries 5.1 and 5.2, we obtain the following corollary.

Corollary 5.4. *If every matrix A_i, $i = 1, \ldots, q$, in (2) is normal, then, for asymptotic stability of the system (2), it is necessary and sufficient that the condition (12) be satisfied.*

Note that the conditions of Corollary 5.4 ensure the existence of the Lyapunov function $V(x) = (\|x\|_2)^2 = x^T x$ for the system (2). Note also that the conditions of Corollaries 5.3 and 5.4 are automatically satisfied in the case when matrices A_i, $i = 1, \ldots, q$, in (2) are diagonal and all their diagonal entries have magnitudes strictly less than one.

Based on Theorem 5.1, one can develop a numerical algorithm, similar to that in [4], which is theoretically capable of determining asymptotic stability of the system (2) in a finite number of steps. However, because the number $\hat{s}$ in (24) may be extremely large, this computer-aided test cannot always be implemented under the existing computational capacity. Therefore, it is reasonable to obtain other criteria of asymptotic stability for the system (2) which could be less involved in actual implementations.

Using Lemma 4.2 and the results obtained in [20], [21] by application of Lyapunov's second method [10], [15] to the problems of robust and absolute stability of discrete systems, one can establish criteria for asymptotic stability of the system (2) in the form similar to Theorems 3–5 of [20] and Theorem 1 of [21]. Note that the present results are mostly concerned with singling out the parametric classes of Lyapunov functions defining the necessary and sufficient conditions of asymptotic stability of the system (2). The main result of this paper obtained in the framework of such an approach is stated in the following theorem.

Theorem 5.2. For the asymptotic stability of the discrete system (2), it is necessary and sufficient that for some integer $m \geq n$ there exists a full column rank matrix $H \in R^{m\times n}$ and a constant θ $(0 < \theta < 1)$ such that a piecewise linear Lyapunov function of the polyhedral vector norm type

$$V_H(x) = \|x\|_H = \|Hx\|_\infty \tag{26}$$

satisfies the inequality

$$\max_{1\leq i\leq q} V_H(A_i x) \leq \theta V_H(x) \tag{27}$$

for all $x \in R^n$.

The proof of Theorem 5.2 follows similar steps as those in the proof of Theorem 3 given in [20] with some nonessential modifications and is therefore omitted here.

We note that the positive definiteness of the polyhedral Lyapunov function $V_H(x)$ in (26) follows directly from the rank condition on the matrix H in Theorem 5.2, and the level surfaces of this function are boundaries of a centrally symmetric convex polytope. The inequality (27) guarantees the strict decreasing of the Lyapunov function $V_H(x)$ along the solutions of the system (2), and the constant θ characterizes the rate of decrease.

From Theorem 5.2 and Corollary 5.1, the following corollary can be obtained.

Corollary 5.5. *The system (2) is asymptotically stable if and only if there exists in R^n a polyhedral vector norm $\|x\|_H$ of the type (26), such that for the corresponding induced matrix norm the conditions*

$$\|A_i\|_H < 1, \quad i = 1, \ldots, q \tag{28}$$

are satisfied.

Theorem 5.2 makes it possible to consider for the system (2) Lyapunov functions of the type of even degree $2p$ ($p \geq 1$) homogeneous polynomial

$$V_{H,p}(x) = \sum_{i=1}^{m} (H_i x)^{2p} = (\|Hx\|_{2p})^{2p}, \tag{29}$$

where H_i, $i = 1, \ldots, m$, is the ith row of the matrix H in (26). Under the above-mentioned rank condition, the function $V_{H,p}(x)$ will be a strictly convex positive definite function in the space R^n.

We have the following corollary of Theorem 5.2.

Corollary 5.6. *For the asymptotic stability of the system (2), it is necessary and sufficient that there exist a full column rank matrix $H \in R^{m\times n}$ (with H_i being its ith row, $i = 1, \ldots, m$), a finite integer $p \geq 1$, and a constant θ ($0 < \theta < 1$), such that the Lyapunov function $V_{H,p}(x)$ defined by (29) satisfies the inequality*

$$\max_{1\leq i\leq q} V_{H,p}(A_i x) \leq \theta V_{H,p}(x)$$

for all $x \in R^n$.

The proof of Corollary 5.6 is similar to the proof of Theorem 4 in [20] and is omitted here.

The use of Theorem 5.2 enables us to obtain an algebraic criterion for asymptotic stability of the system (2) in the form of solvability conditions of a set of matrix equations. Such a criterion is stated in the next theorem.

Theorem 5.3. For the asymptotic stability of the system (2), it is necessary and sufficient that there exist a finite integer $m \geq n$, a full column rank matrix $H \in R^{m\times n}$, and $m \times m$ matrices $\Gamma_i \in R^{m\times m}$, $i = 1, \ldots, q$, satisfying the conditions

$$\|\Gamma_i\|_\infty < 1, \quad i = 1, \ldots, q \tag{30}$$

such that the matrix relations

$$HA_i = \Gamma_i H, \quad i = 1, \ldots, q \tag{31}$$

are satisfied.

The proof of Theorem 5.3 follows from the proof of Theorem 5 in [20] with some minor modifications and is omitted here.

Note that the conditions of Theorem 5.3 are automatically satisfied if $\|A_i\|_\infty < 1$ for every matrix A_i, $i = 1, \ldots, q$, in (2). In this case, $m = n$, $H = I_n$, $\Gamma_i = A_i$, $i = 1, \ldots, q$, and $V_H(x) = \|x\|_\infty$ (see also Corollaries 5.1 and 5.5).

Note also that the matrix relations (31) together with the conditions (30) are equivalent to the conditions (28) for the matrix norm induced by the Lyapunov function $V_H(x) = \|x\|_H$ in (26). As a consequence, we can choose a θ in (27) as

$$\theta = \max_{1 \le i \le q} \{\|\Gamma_i\|_\infty\}.$$

Thus, Theorem 5.2 reduces the problem of asymptotic stability for the system (2) to the problem of existence of a polyhedral vector norm type Lyapunov function (26). In turn, Theorem 5.3 reduces the problem of the construction of such a Lyapunov function to the problem of solvability of the matrix relations (31), which can be considered as a set of matrix equations in terms of unknown matrices $H \in R^{m\times n}$ and $\Gamma_i \in R^{m\times m}$, $i = 1, \ldots, q$.

In the general case, the problem of solving matrix equations (31) can effectively be realized only by numerical methods. The main difficulty in finding the unknown matrices $H \in R^{m\times n}$ and $\Gamma_i \in R^{m\times m}$, $i = 1, \ldots, q$, satisfying the matrix equations (31) is that the integer m can be very large ($m \gg n$), and it should be chosen experimentally in practical numerical calculations by incrementing its value. With the value of m being fixed (for example, $m = n$), the conditions of Theorems 5.2 and 5.3 become only sufficient for asymptotic stability of the system (2). In this case, for solving matrix equations (31), one can use a numerical algorithm developed in [26] with the use of the linear programming method and the idea of scaling. The improvement of sufficient conditions of asymptotic stability for the system (2) obtained in this way can be achieved at the expense of increasing the value of the integer parameter m. Some simpler sufficient conditions of asymptotic stability of the system (2), which can coincide with necessary and sufficient conditions in special cases, are presented in Section 6.

6. Majorant approach to the problem of asymptotic stability

In this section, following [3], [22], [29], [30], we establish a simple sufficient condition for asymptotic stability of the system (2). The proposed condition is a special corollary of the results established in Section 5 and is relatively simple to implement. It is shown that this condition becomes necessary and sufficient for some special cases of the system (2).

Let $|A|$ denote the matrix obtained from the matrix A by taking the absolute value of all its entries, i.e. if $A = [a_{ij}] \in R^{m\times n}$, then $|A| = [|a_{ij}|] \in R^{m\times n}$. The inequality $A \ge 0$ (or $A \le 0$) is understood to be element-wise, and 0 is used to denote the matrix of corresponding dimension whose entries are all equal to zero. For two matrices, $A \in R^{m\times n}$ and $B \in R^{m\times n}$, the inequality $B \geq A$ (or $A \leq B$) is equivalent to the inequality $B - A \geq 0$ (or $A - B \leq 0$).

Let us introduce a nonnegative majorant matrix

$$\hat{A} = \max_{1\leq i\leq q} \{|A_i|\}, \tag{32}$$

where the maximum is understood to be element-wise.

We are now in a position to establish the following result.

Theorem 6.1. *The discrete system (2) is asymptotically stable if the matrix $\hat{A}$ in (32) is Schur stable, i.e. the condition*

$$\rho(\hat{A}) < 1 \tag{33}$$

holds.

Proof. Because $\hat{A} \geq 0$, from (33) and Lemma 3.1 in [14] it follows that there exists a positive definite diagonal matrix $H \in R^{n\times n}$,

$$H = \operatorname{diag}\{h_1, \ldots, h_n\},\ h_i > 0,\ i = 1, \ldots, n \tag{34}$$

such that $\|H\hat{A}H^{-1}\|_\infty < 1$. Let us denote $\Gamma = H\hat{A}H^{-1}$. Obviously, $H\hat{A} = \Gamma H$, rank $H = n$, $\Gamma \geq 0$, and $\|\Gamma\|_\infty < 1$.

Let us denote $\Gamma_i = HA_iH^{-1}$, $i = 1, 2, \ldots, q$. It is easily verified that $|\Gamma_i| = H|A_i|H^{-1} \leq \Gamma$, and hence $\|\Gamma_i\|_\infty \leq \|\Gamma\|_\infty < 1$. Because $HA_i = \Gamma_i H$, $i = 1, 2, \ldots, q$, then in this case the conditions (30) are satisfied, and the matrix relations (31) are fulfilled with diagonal matrix H defined by (34). This completes the proof of Theorem 6.1. □

Note that Theorem 6.1 ensures the existence for the system (2) of the Lyapunov function of the form [14], [22]

$$V_h(x) = \max_{1\leq i\leq n} \{h_i|x_i|\},\ h_i > 0,\ i = 1, \ldots, n,$$

which is a special case of the Lyapunov function $V_H(x)$ in (26) corresponding to the diagonal matrix $H \in R^{n\times n}$ in (34). In this case, we can choose

$$\theta = \|\Gamma\|_\infty = \|H\hat{A}H^{-1}\|$$

in the inequality (27).

The condition (33) of Theorem 6.1 is only sufficient for asymptotic stability of the system (2). However, for special cases of system (2) satisfying some additional conditions, Theorem 6.1 becomes a necessary and sufficient condition of asymptotic stability.

To formulate the next theorem, we recall that from [29], a matrix $A \in R^{n\times n}$ is called a Morishima matrix if by symmetric row and column permutations it can be transformed into the block form

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{11} \geq 0$, $A_{22} \geq 0$ are square submatrices and $A_{12} \leq 0$, $A_{21} \leq 0$. A matrix $A \in R^{n\times n}$ is a Morishima matrix if and only if $WAW^{-1} = |A|$ for some diagonal matrix $W = \text{diag}\{w_1, \ldots, w_n\}$ with diagonal entries $w_i = \pm 1$, $i = 1, \ldots, n$. If A is a Morishima matrix, then it is Schur stable $[\rho(A) < 1]$ if and only if $|A|$ is Schur stable $[\rho(|A|) < 1]$.

Our next result singles out several classes of system (2) whose asymptotic stability is equivalent to Schur stability of a majorant matrix $\hat{A}$ in (32).

Theorem 6.2. *The sufficient condition (33) in Theorem 6.1 is also necessary for asymptotic stability of the system (2) in each of the following cases.*

(1) *Among the given matrices A_i, $i = 1, \ldots, q$, in (2), there exists at least one matrix $A_{\hat{i}}$ such that either $A_{\hat{i}} = \hat{A}$ or $A_{\hat{i}} = -\hat{A}$.*

(2) *Among the given matrices A_i, $i = 1, \ldots, q$, in (2), there exists at least one matrix $A_{\hat{i}}$ such that $|A_{\hat{i}}| = \hat{A}$, and either $A_{\hat{i}}$ or $-A_{\hat{i}}$ is a Morishima matrix.*

(3) *All matrices A_i, $i = 1, \ldots, q$, in (2) are either all upper triangular or all lower triangular.*

Proof. In case 1, the proof follows directly from Corollary 5.2 and the obvious relation $\rho(-\hat{A}) = \rho(\hat{A}) = \rho(A_{\hat{i}}) < 1$.

In case 2, from Corollary 5.2, it follows that the matrix $A_{\hat{i}}$ (and the matrix $-A_{\hat{i}}$) is Schur stable. Because either $A_{\hat{i}}$ or $-A_{\hat{i}}$ is a Morishima matrix, then the matrix $|A_{\hat{i}}| = \hat{A}$ is also Schur stable.

In case 3, if all matrices A_i, $i = 1, \ldots, q$, in (2) are upper triangular, then, from the definition of the matrix $\hat{A}$ in (32), it follows that $\hat{A}$ is also upper triangular and

$$\rho(\hat{A}) = \max_{1\leq i\leq q} \{\rho(A_i)\}. \tag{35}$$

From (12) and (35), we have (33), i.e. the matrix $\hat{A}$ is Schur stable. In the case of lower triangular matrices A_i, $i = 1, \ldots, q$, the proof is fully analogous.

This completes the proof of Theorem 6.2. □

Note that in the case of upper or lower triangular matrices A_i, $i = 1, \ldots, q$, in (2), the asymptotic stability of the system (2) depends only on their diagonal elements. Therefore, the off-diagonal elements have no effect on asymptotic stability of the system (2) in this case. Note also in conclusion that in all three cases given by Theorem 6.2, the Schur stability of the majorant matrix $\hat{A}$ follows immediately from the condition (12). Therefore, in addition to the cases considered in Corollaries 5.3 and 5.4, Theorem 6.2 gives three other cases when the condition (12) is necessary and sufficient for asymptotic stability of the system (2).

7. Conclusions

In this paper, we have investigated the problem of asymptotic stability for a class of linear shift-invariant discrete systems with multiple independent variables. We have established the equivalence of this problem to the problem of robust stability of a special ordinary linear shift-varying discrete system with matrix uncertainty set defined by the coefficient matrices of the original discrete system. On the basis of this equivalence, using the variational method and Lyapunov's second method, necessary and sufficient conditions of asymptotic stability have been obtained in different forms for the given class of discrete systems. A simple sufficient condition of asymptotic stability has been provided which becomes necessary and sufficient for several special cases of the discrete system under consideration.

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