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Criteria for robust absolute stability of time-varying nonlinear continuous-time systems☆

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Abstract

The present paper establishes results for the robust absolute stability of a class of nonlinear continuous-time systems with time-varying matrix uncertainties of polyhedral type and multiple time-varying sector nonlinearities. By using the variational method and the Lyapunov Second Method, criteria for robust absolute stability are obtained in different forms for the given class of systems. Specifically, the parametric classes of Lyapunov functions are determined which define the necessary and sufficient conditions of robust absolute stability. The piecewise linear Lyapunov functions of the infinity vector norm type are applied to derive an algebraic criterion for robust absolute stability in the form of solvability conditions of a set of matrix equations. Several simple sufficient conditions of robust absolute stability are given which become necessary and sufficient for special cases. Two examples are presented as applications of the present results to a particular second-order system and to a specific class of systems with time-varying interval matrices in the linear part. © 2002 Published by Elsevier Science Ltd.

Keywords: Continuous-time systems; Time-varying systems; Lyapunov methods; Robust stability; Absolute stability; Differential inclusion; Variational method

1. Introduction

The problem of robust control and robust stability of uncertain dynamic systems with parametric or nonparametric uncertainties has been the subject of considerable research efforts in the past two decades. Many significant results covering these issues have been reported in the literature (see, e.g., Barmish, 1994; Bhattacharyya, Chapellat, & Keel, 1995; Blanchini & Miani, 1999; Chen & Leitmann, 1987; Dorato, Tempo, & Muscato, 1993; Dorato & Yedavalli, 1990; Freeman & Kokotovic, 1996; Hinrichsen, Ilchmann, & Pritchard, 1989; Kaszkurewicz & Bhaya, 1993; Kharitonov, 1979; Luo, Zhang, & Johnson, 1994; Molchanov & Bauer, 1999; Sezer & Šiljak, 1994; Wang & Michel, 1996; Wang, Michel, & Liu, 1994; Yedavalli & Liang, 1986; Zhou & Khargonekar, 1987). On the other hand, the classic problem of absolute stability of a class of nonlinear control systems with a fixed matrix in the linear part of the system and one or multiple uncertain nonlinearities satisfying the sector constraints has been extensively studied (Aizerman & Gantmacher, 1964; Khalil, 1996; Luré, 1957; Molchanov & Pyatnitskiy, 1986; Narendra & Taylor, 1973; Popov, 1973; Pyatnitskii, 1970; Pyatnitskiy & Rapoport, 1996; Rapoport, 1994; Savkin & Petersen, 1995; Yakubovich, 1967) along with the publication of the initial work of Kharitonov (1979) which laid the foundation for the problem of robust stability. Meanwhile, in the light of modern robustness terminology, absolute stability can be considered as the robust global asymptotic stability with respect to variations (or changes) of nonlinearities from a given class.

There has been some recent works devoted to the investigation of the more general problem of *robust absolute stability* of nonlinear control systems with uncertainties both in the linear part and in the nonlinear part of the system (Bhattacharyya et al., 1995; Grujić & Petkovski, 1987; Tesi

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& Vicino, 1991; Tsypkin & Polyak, 1993). To the best of our knowledge, in all the papers on this problem, only sufficient conditions of robust absolute stability were obtained based mainly on the well-known circle and Popov criteria. Thus, at present, the problem of obtaining *necessary and sufficient* conditions of robust absolute stability is of great theoretical and practical interest.

In our recent work (Liu & Molchanov, 2001), we have established necessary and sufficient conditions of robust absolute stability for a class of nonlinear *discrete-time* systems with time-varying matrix uncertainty of polyhedral type in the linear part and multiple time-varying sector nonlinearities in the nonlinear part of the system. It is the purpose of the present paper to extend the results obtained in Liu and Molchanov (2001) to *continuous-time* case, i.e., to establish necessary and sufficient conditions of *robust absolute stability* for a class of nonlinear continuous-time systems with time-varying matrix uncertainty of polyhedral type and multiple time-varying sector nonlinearities. Using the variational method developed in Pyatnitskii (1970, 1975) for a class of nonlinear continuous-time systems and the Lyapunov Second Method (cf. Hahn, 1967; Khalil, 1996; Molchanov & Pyatnitskiy, 1986, 1989), we establish criteria for the robust absolute stability of the class of systems under consideration. The parametric classes of Lyapunov functions which define the necessary and sufficient conditions of robust absolute stability of such systems are identified. An algebraic criterion for robust absolute stability in the form of solvability conditions of a set of matrix equations is obtained using Lyapunov functions from a class of piecewise linear functions of the infinity vector norm type.

In general, the main problem related to the implementation of the obtained criteria of robust absolute stability is their computational complexity. Therefore, following Daoyi (1985), Kaszkurewicz and Bhaya (1993), Sezer and Šiljak (1994), Wang and Michel (1996), Wang, Michel, and Liu (1994), we obtain several simple and computationally efficient sufficient conditions for robust absolute stability of the class of continuous-time systems considered herein. We will indicate that these conditions become necessary and sufficient for a few special cases. We will conclude the paper with two examples of application of these results to a particular second-order system and to a special class of systems with time-varying interval matrices in the linear part.

2. Preliminaries

Let R denote the set of real numbers. Let R^+ denote the set of nonnegative real numbers, i.e., $R^+ = [0, \infty)$. Let N denote the set of positive integers, i.e., $N = \{1, 2, \dots\}$. Let R^n denote real n-space. If $x \in R^n$, then $x' = [x_1, \dots, x_n]$ denotes the transpose of x. Let $R^{m\times n}$ denote the set of $m \times n$ real matrices. If $A = [a_{ij}] \in R^{m\times n}$, then A' denotes the transpose of A. We let $\|x\|$ denote any one of the equivalent vector norms on R^n. In particular, the l_p norms $\|x\|_p$, $1 \leqslant p \leqslant \infty$, are defined by

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}, \quad 1 \leqslant p < \infty$$

and $\|x\|_\infty = \max_{1\leqslant i\leqslant n} |x_i|$. The matrix norm $\|A\|$, defined on $R^{n\times n}$, and induced by the vector norm $\|x\|$ in R^n, are defined as $\|A\| = \max_{\|x\|=1} \|Ax\|$. In particular, we have

$$\|A\|_\infty = \max_{\|x\|_\infty=1} \|Ax\|_\infty = \max_{1\leqslant i\leqslant n} \sum_{j=1}^{n} |a_{ij}|.$$

The matrix measure $\mu(A)$ (Axelsson, 1994; Coppel, 1965; Desoer & Vidyasagar, 1975; Vidyasagar, 1978) is defined by

$$\mu(A) = \lim_{\varepsilon\to 0^+} \varepsilon^{-1}(\|I_n + \varepsilon A\| - 1) = \lim_{\varepsilon\to 0^+} \varepsilon^{-1} \ln(\|e^{\varepsilon A}\|)$$

where I_n is the $n \times n$ identity matrix. In particular, the infinity norm $\|A\|_\infty$ induces the matrix measure

$$\mu_\infty(A) = \max_{1\leqslant i\leqslant n} \left\{ a_{ii} + \sum_{j=1, j\neq i}^{n} |a_{ij}| \right\}.$$

We consider the following class of nonlinear continuous-time systems described by the equation

$$\dot{x} = A(t)x + \sum_{j=1}^{r} b_j \phi_j(\sigma, t), \tag{1}$$

where $x \in R^n$, $t \in R^+$, $\sigma' = [\sigma_1, \dots, \sigma_r]$ with $\sigma_j = c_j' x$, and $b_j \in R^n$ and $c_j \in R^n$ are constant vectors for $j = 1, \dots, r$. It is assumed that the time-varying matrix function $A(t)$ in (1) is Lebesgue measurable on R^+ and its values belong to a given polytope of matrices almost everywhere on R^+, i.e.,

$$A(t) \in co\{A_1, \dots, A_q\} \subset R^{n\times n}, \tag{2}$$

where $A_1, \dots, A_q$ are fixed matrices and $co\{\cdot\}$ denote the convex hull of a set. We let $\mathscr{A}$ denote the set of all such time-varying matrix functions $A(t)$. The matrix polytope $co\{A_1, \dots, A_q\}$ describes structured parametric uncertainty in the linear part of the system (1). We assume that the uncertain time-varying nonlinear functions $\phi_j(\sigma, t)$, $j = 1, \dots, r$, defined for all $\sigma \in R^r$ and $t \in R^+$, are continuous in $\sigma \in R^r$ and Lebesgue measurable in $t \in R^+$, i.e., they are Caratheodory functions (Coddington & Levinson, 1955), and satisfy the conditions $\phi_j(0, t) \equiv 0$ for $j = 1, \dots, r$ and $t \in R^+$. We also assume the sector constraints

$$\alpha_j \sigma_j^2 \leqslant \phi_j(\sigma, t)\sigma_j \leqslant \beta_j \sigma_j^2, \quad j = 1, \dots, r, \tag{3}$$

where $\alpha_j \in R$ and $\beta_j \in R$, $j = 1, \dots, r$, are given constants. We use Φ to denote the set of all such time-varying nonlinear vector-functions $\phi(\sigma, t)$, where $\phi'(\sigma, t) = [\phi_1(\sigma, t), \dots, \phi_r(\sigma, t)]$. We note that the above-mentioned conditions on the matrix function $A(t) \in \mathscr{A}$ and nonlinear vector-function $\phi(\sigma, t) \in \Phi$ ensures the existence of an absolutely continuous solution of system (1) for any initial time $t_0 \in R^+$ and any initial state $x_0 \in R^n$, which is defined for all $t \geqslant t_0$ and satisfies the equation (1) almost everywhere on $[t_0, \infty)$. Thus, solution $x_{A,\phi}(t,t_0,x_0)$ of the system (1) is defined by an arbitrary choice of a time-varying matrix $A(t)$ from the set $\mathscr{A}$ and a vector nonlinearity $\phi(\sigma,t)$ from the set Φ, in addition to an initial point (t_0,x_0).

Note that due to the fact that $\phi(0,t) \equiv 0$, we have $x_{A,\phi}(t,t_0,0) \equiv 0$, $t \geqslant t_0$, for any matrix function $A(t) \in \mathscr{A}$ and any nonlinearity $\phi(\sigma,t) \in \Phi$. Therefore, we will use the notation $x(t) \equiv 0$ for the zero solution of (1).

Similar to the definition given in Liu and Molchanov (2001), in this paper, the robust absolute stability of the system (1) will be considered in the sense of the following definition.

Definition 2.1. The system (1) is said to be robustly absolutely stable with respect to the set $\mathscr{A} \times \Phi$ defined by (2) and (3) if its zero solution $x(t) \equiv 0$ is globally asymptotically stable for any time-varying matrix $A(t) \in \mathscr{A}$ and any vector nonlinearity $\phi(\sigma,t) \in \Phi$.

We note that if there is no uncertainty in the linear part of the system (1), that is, if all matrices A_i in (2) are identical (i.e., $A_i \equiv A, i = 1,\dots,q$) and the set $\mathscr{A}$ degenerates to the "singleton" or "point" A in the matrix space $R^{n\times n}$, then the problem of robust absolute stability becomes that of absolute stability of the system (1) with respect to the set Φ of time-varying vector nonlinearities $\phi(\sigma,t)$ defined by (3). This problem has been previously studied by many authors including Khalil (1996), Molchanov and Pyatnitskiy (1986), Narendra and Taylor (1973), Popov (1973), Pyatnitskii (1970), Pyatnitskiy and Rapoport (1996), Rapoport (1994), Savkin and Petersen (1995), and Yakubovich (1967). On the other hand, if $\phi(\sigma,t) \equiv 0$, the problem of robust absolute stability reduces to that of robust stability of the linear time-varying system

$$\dot{x} = A(t)x \tag{4}$$

with respect to the set $\mathscr{A}$ defined by (2), which was considered in many previous works (see, e.g., Hinrichsen et al., 1989; Kaszkurewicz & Bhaya, 1993; Luo et al., 1994; Molchanov & Bauer, 1999; Molchanov & Pyatnitskiy, 1989; Pyatnitskii, 1975; Yedavalli & Liang, 1986).

The main goal of this work is to obtain necessary and sufficient conditions for robust absolute stability of the system (1) with respect to the set $\mathscr{A} \times \Phi$ defined by (2) and (3). Our main results are given in Sections 4 and 5.

3. Reduction to robust stability of linear systems

In this section, we consider along with nonlinear system (1) the linear time-varying system

$$\dot{x} = \left[A(t) + \sum_{j=1}^{r} u_j(t) b_j c_j' \right] x, \tag{5}$$

where $A(t) \in \mathscr{A}$ and $u_j(t)$, $j = 1,\dots,r$, are arbitrary Lebesgue measurable functions on R^+ satisfying

$$\alpha_j \leqslant u_j(t) \leqslant \beta_j, \quad j = 1,\dots,r \tag{6}$$

for almost all $t \in R^+$. The set of such vector-function $u(t)$, where $u'(t) = [u_1(t),\dots,u_r(t)]$, is denoted by U.

The system (5) can be obtained from the system (1) by considering the functions $\phi_j(\sigma,t)$ of particular type given by $\phi_j(\sigma,t) = u_j(t)\sigma_j$, $j = 1,\dots,r$, which form a subset of the set Φ. In this case, the inequalities (6) are a direct consequence of inequalities (3). *Robust absolute stability* of the system (5) with respect to the set $\mathscr{A} \times U$ will be understood in the sense of Definition 2.1 given in Section 2, replacing Φ by U and $\phi(\sigma,t) \in \Phi$ by $u(t) \in U$.

It follows from Molchanov and Pyatnitskiy (1986, 1989), on the basis of results presented in Aubin and Cellina (1984) and Filippov (1988) that the nonlinear system (1) on the set $\mathscr{A} \times \Phi$ and the linear system (5) on the set $\mathscr{A} \times U$ are equivalent to the following time-invariant differential inclusion

$$\dot{x} \in F(x), \tag{7}$$

where the multivalued vector-function $F(x)$ is defined for all $x \in R^n$ by

$$F(x) = \left\{ y\colon\ y = \left(\sum_{i=1}^{q} \lambda_i A_i + \sum_{j=1}^{r} \gamma_j b_j c_j' \right) x, \right.$$
$$\lambda_i \geqslant 0, i = 1,\dots,q; \sum_{j=1}^{q} \lambda_i = 1;$$
$$\left. \alpha_j \leqslant \gamma_j \leqslant \beta_j,\ j = 1,\dots,r \right\}. \tag{8}$$

The equivalence is regarded in the sense of coincidence of the sets of absolutely continuous solutions of the system (1) [for all admissible $A(t) \in \mathscr{A}$, $\phi(\sigma,t) \in \Phi$], of the system (5) [for all admissible $A(t) \in \mathscr{A}$, $u(t) \in U$], and of the differential inclusion (7) and (8). Therefore, the problem of robust absolute stability of the nonlinear system (1) with respect to the set $\mathscr{A} \times \Phi$ is equivalent to a similar problem for the linear system (5) with respect to the set $\mathscr{A} \times U$, and both problems can be reduced to the problem of global asymptotic stability of the zero solution $x(t) \equiv 0$ of the differential inclusion (7) and(8).

As a result, we obtain the following lemma.

Lemma 3.1. *For the robust absolute stability of system* (1) *with respect to the set $\mathscr{A} \times \Phi$, it is necessary and sufficient that system* (5) *is robustly absolutely stable with respect to the set $\mathscr{A} \times U$.*

We introduce into consideration the matrices

$$\tilde{A}_{iv} = A_i + \sum_{j=1}^{r} \tilde{\gamma}_{jv} b_j c_j',$$
$$i = 1,\dots,q;\ v = 1,\dots,s;\ s = 2^r, \tag{9}$$

where the parameters $\tilde{\gamma}_{jv}$ can independently take only the extreme values $\tilde{\gamma}_{jv}=\alpha_j$ or $\tilde{\gamma}_{jv}=\beta_j$, for $j=1,\dots,r$. It can easily be seen that any vector $y\in F(x)$ in (8) admits an equivalent representation as $y=Ax$, where the matrix $A\in R^{n\times n}$ belongs to the matrix polytope $co\{\tilde{A}_{11},\dots,\tilde{A}_{1s};\cdots;\tilde{A}_{q1},\dots,\tilde{A}_{qs}\}\subset R^{n\times n}$. We use $\tilde{\mathscr{A}}$ to denote the set of all Lebesgue measurable matrix functions $A(t)$:$R^+\to R^{n\times n}$ such that for almost every fixed $t\in R^+$,

$$A(t)\in co\{\tilde{A}_{11},\dots,\tilde{A}_{1s};\dots;\tilde{A}_{q1},\dots,\tilde{A}_{qs}\}. \tag{10}$$

Then, in accordance with Lemma 3.1, the problem of robust absolute stability of the system (1) with respect to the set $\mathscr{A}\times\Phi$ reduces to an equivalent problem of robust stability of the linear time-varying system (4) with respect to the set $\tilde{\mathscr{A}}$ defined by (9) and (10) in the sense of the following definition in Liu and Molchanov (2001) and Molchanov and Bauer (1999).

Definition 3.1. The system (4) is said to be robustly stable with respect to the set $\tilde{\mathscr{A}}$ if the zero solution $x(t)\equiv 0$ of this system is globally asymptotically stable for any time-varying matrix $A(t)\in\tilde{\mathscr{A}}$.

Note that for linear system (4) the concept of *global* asymptotic stability is equivalent to that of *local* asymptotic stability (Khalil, 1996). For the same reason, robust stability of the system (4) with respect to the set $\tilde{\mathscr{A}}$ is equivalent to the condition

$$\lim_{t\to\infty}\|x_A(t,t_0,x_0)\|=0 \tag{11}$$

for any solution $x_A(t,t_0,x_0)$ of the system (4). Note also that, it follows from the results obtained in Filippov (1988) and Lasota and Strauss (1971), if the limit condition (11) is valid, then there exist constants $L\geqslant 1$ and $\xi>0$ such that the estimate

$$\|x_A(t,t_0,x_0)\|\leqslant L\|x_0\|\mathrm{e}^{-\xi(t-t_0)},\quad t\geqslant t_0 \tag{12}$$

is fulfilled for any $t_0\in R^+$, any $x_0\in R^n$ and any matrix function $A(t)\in\tilde{\mathscr{A}}$. The opposite assertion is obvious. Thus, the following statement holds.

Lemma 3.2. *The system* (1) *is robustly absolutely stable with respect to the set* $\mathscr{A}\times\Phi$ *if and only if the system* (4) *is globally exponentially stable* [*in the sense of* (12)] *with respect to the set* $\tilde{\mathscr{A}}$ *defined by* (9) *and* (10).

Lemma 3.2 is the starting point for obtaining necessary and sufficient conditions of robust absolute stability presented in Section 4.

4. Main results

In this section, we will derive necessary and sufficient conditions for robust absolute stability of the system (1) with respect to the set $\mathscr{A}\times\Phi$. We will employ the variational method (Pyatnitskii, 1975) and the Lyapunov Second Method (Hahn, 1967; Khalil, 1996), which were also used in Molchanov and Pyatnitskiy (1986, 1989), Pyatnitskii (1970) for the problem of absolute stability of nonlinear continuous-time control systems with fixed time-invariant matrix in the linear part [i.e., when $A_i\equiv A, i=1,\dots,q$, in (2)].

First, we present a criterion for robust absolute stability that is obtained by using the variational method. For any $T\in R^+$ and any $k\in N$, we use $\Pi_k(T)$ to denote the set of all matrix products $\pi_k(T)$ defined by

$$\pi_k(T)=\mathrm{e}^{\tilde{A}_{i_kv_k}\tau_{i_kv_k}}\cdots\mathrm{e}^{\tilde{A}_{i_1v_1}\tau_{i_1v_1}}, \tag{13}$$

where

$$\sum_{j=1}^{k}\tau_{i_jv_j}=T,\quad \tau_{i_jv_j}\in R^+,\ \ i_j\in\{1,\dots,q\},$$
$$v_j\in\{1,\dots,s\},\ \ j=1,2,\dots,k. \tag{14}$$

We are now in a position to establish the following result.

Theorem 4.1. *The system* (1) *is robustly absolutely stable with respect to the set* $\mathscr{A}\times\Phi$ *if and only if there exists a* $\bar{T}>0$, *such that*

$$\sup_{k\in N}\ \sup_{\pi_k(\bar{T})\in\Pi_k(\bar{T})}\{\|\pi_k(\bar{T})\|\}<1. \tag{15}$$

Proof. Because the differential inclusion (7) is time-invariant, we assume that $t_0=0$ and consider the compact reachability set $\mathscr{R}(T,x_0)$ of the system (4) from the state $x_0\in R^n$ at time T. The set $\mathscr{R}(T,x_0)$ consists of all $x\in R^n$ such that there exist matrix functions $A(t)\in\tilde{\mathscr{A}}$ and corresponding solutions $x_A(t,x_0)$ of the system (4) with the properties that $x_A(0,x_0)=x_0$ and $x_A(T,x_0)=X_A(T)x_0=x$, where $X_A(t)$ is the fundamental matrix of the equation (4) generated by matrix $A(t)$ with $X_A(0)=I_n$.

We define the functions

$$G(T,x_0)=\max_{x\in\mathscr{R}(T,x_0)}\|x\| \tag{16}$$

and

$$f(T)=\max_{\|x_0\|=1}G(T,x_0). \tag{17}$$

We note that by virtue of the linearity of the system (4) and the semi-group property

$$\mathscr{R}(T_1+T_2,x_0)=\mathscr{R}(T_2,\mathscr{R}(T_1,x_0))=\bigcup_{x\in\mathscr{R}(T_1,x_0)}\mathscr{R}(T_2,x),$$

the relation

$$G(T_1+T_2,x_0)=\max_{x\in\mathscr{R}(T_1,x_0)}G(T_2,x) \tag{18}$$

and the inequality

$$f(T_1+T_2)\leqslant f(T_1)f(T_2) \tag{19}$$

are valid for any $x_0 \in R^n$, $T_1 \in R^+$, and $T_2 \in R^+$. Similar to the proof of Theorem 3.9 in Khalil (1996) and Theorem 2 in Pyatnitskii (1975), we show that

$$f(\bar{T}) < 1 \tag{20}$$

for some $\bar{T} > 0$ is necessary and sufficient for the global exponential stability of the system (4) with respect to the set $\tilde{\mathscr{A}}$, and therefore, by Lemma 3.2, for robust absolute stability of the system (1) with respect to the set $\mathscr{A} \times \Phi$.

Indeed, using the estimate (12) with $t_0 = 0$, $t = T$, and relation (17), we obtain the inequality $f(T) \leqslant L\mathrm{e}^{-\xi T}$ from which it follows that condition (20) is true when $\bar{T} > \xi^{-1}\ln L$. This proves necessity of condition (20).

To prove sufficiency, we note that by the linearity of the system (4) its solutions satisfy the inequality

$$\|x_A(t,t_0,x_0)\| \leqslant f(t-t_0)\|x_0\|, \quad t \geqslant t_0. \tag{21}$$

Let $t-t_0 = l\bar{T}+\tau$, where $0 \leqslant \tau < \bar{T}$, $l=[(t-t_0)/\bar{T}]$ and $[y]$ denotes the integer part of $y \in R$. It follows from (19) that

$$\begin{aligned} f(t-t_0) &\leqslant f(\tau)f(l\bar{T}) \leqslant f(\tau)f^l(\bar{T}) \\ &\leqslant f_0 f^{-1}(\bar{T})(f(\bar{T}))^{(t-t_0)/\bar{T}}, \end{aligned} \tag{22}$$

where $f_0 = \max_{0 \leqslant \tau \leqslant \bar{T}} f(\tau) \geqslant 1$ and $f^{-1}(\bar{T}) = 1/f(\bar{T})$.

From (20)–(22), we have the exponential estimate (12) with $L = f_0 f^{-1}(\bar{T}) > 1$ and $\xi = -\bar{T}^{-1}\ln f(\bar{T}) > 0$. This proves the sufficiency of the condition (20).

Let $\Omega(T)$ be the set of all fundamental matrices $X_A(T)$ for the set $\tilde{\mathscr{A}}$ of matrix functions $A(t)$. Let $\Omega_0(T)$ be the similarly defined set for piecewise constant matrix functions $A(t)$ that take only the vertex values $\tilde{A}_{iv}$, $i=1,2,\ldots,q$; $v=1,2,\ldots,s$; $s=2^r$, and have a finite number of switching points on each finite interval of the set R^+. Then, according to Sussmann (1972), $\Omega_0(T)$ is dense in $\Omega(T)$, i.e., the closure of $\Omega_0(T)$ is equal to $\Omega(T)$. But for piecewise constant matrix function $A(t)$ on $[0,T]$ with at most $k-1$ switchings the corresponding fundamental matrix $X_A(T)$ belongs to the set $\Pi_k(T)$ defined by (13) and (14). Combining these reasonings, we get

$$\begin{aligned} f(T) &= \max_{X_A(T)\in\Omega(T)} \{\|X_A(T)\|\} \\ &= \sup_{X_A(T)\in\Omega_0(T)} \{\|X_A(T)\|\} \\ &= \sup_{k\in N} \sup_{\pi_k(T)\in\Pi_k(T)} \{\|\pi_k(T)\|\}. \end{aligned} \tag{23}$$

Hence, the inequality (20) is equivalent to the condition (15), which proves the present theorem. □

From Theorem 4.1, the following corollary can easily be obtained.

Corollary 4.1. *The system* (1) *is robustly absolutely stable with respect to the set* $\mathscr{A} \times \Phi$ *if there exists in* R^n *a vector norm* $\|\cdot\|$ *such that the related matrix measure* $\mu(\tilde{A}_{iv}) < 0$ *for any matrix* $\tilde{A}_{iv}$, $i=1,\ldots,q$; $v=1,\ldots,s$; $s=2^r$ *defined by* (9).

Corollary 4.1 is obvious because, due to the well-known inequality (Coppel, 1965; Desoer & Vidyasagar, 1975) $\|\mathrm{e}^{A\tau}\| \leqslant \mathrm{e}^{\mu(A)\tau}$, we have the inequality

$$\|\pi_k(T)\| \leqslant \prod_{j=1}^{k} \|\mathrm{e}^{\tilde{A}_{i_jv_j}\tau_{i_jv_j}}\| \leqslant \mathrm{e}^{\sum_{j=1}^{k} \mu(\tilde{A}_{i_jv_j})\tau_{i_jv_j}} \leqslant \mathrm{e}^{\bar{\mu}T},$$

where $\bar{\mu} = \max_{1\leqslant i\leqslant q, 1\leqslant v\leqslant s}\{\mu(\tilde{A}_{iv})\}$. Under the condition of Corollary 4.1, $\bar{\mu} < 0$, and we will have the inequality (15) for any $\bar{T} > 0$.

Let $\chi(A) = \max_{1\leqslant i\leqslant n}\{Re\lambda_i(A)\}$ where $\lambda_i(A)$, $i=1,\ldots,n$, are eigenvalues of the matrix A (Axelsson, 1994). Recall that an $n \times n$ matrix A is said to be Hurwitz stable if $\chi(A) < 0$. It is well known that the condition $\chi(A) < 0$ is equivalent to the existence of a finite $\bar{T} > 0$, such that $\|\mathrm{e}^{A\bar{T}}\| < 1$. Since $\mathrm{e}^{\tilde{A}_{iv}\bar{T}} \in \Pi_1(\bar{T})$ for any matrix $\tilde{A}_{iv}$ of the form (9), it follows from (15) that $\|\mathrm{e}^{\tilde{A}_{iv}\bar{T}}\| < 1$ if system (1) is robustly absolutely stable. As a result, we obtain the following corollary.

Corollary 4.2. *For the robust absolute stability of the system* (1) *with respect to the set* $\mathscr{A} \times \Phi$, *it is necessary that every vertex matrix* $\tilde{A}_{iv}$ *defined by* (9) *is Hurwitz stable.*

The next two corollaries represent special cases when Hurwitz stability of the vertex matrices $\tilde{A}_{iv}$ is not only necessary but also sufficient for robust absolute stability of the system (1). Other special cases are given in Theorem 5.2 in Section 5.

Corollary 4.3. *If the vertex matrices* $\tilde{A}_{iv}, i=1,\ldots,q$; $v=1,\ldots,s$; $s=2^r$ *defined by* (9) *are pairwise commutative, i.e., if* $\tilde{A}_{iv}\tilde{A}_{jl} = \tilde{A}_{jl}\tilde{A}_{iv}$ *for all* $i,j=1,\ldots,q$; $v,l=1,\ldots,s$; *then for robust absolute stability of the system* (1) *with respect to the set* $\mathscr{A} \times \Phi$, *it is necessary and sufficient that the matrices* $\tilde{A}_{iv}, i=1,\ldots,q$; $v=1,\ldots,s$; $s=2^r$, *are Hurwitz stable.*

Proof. Necessity follows from Corollary 4.2. For sufficiency, let us note that in the case of pairwise commutative matrices $\tilde{A}_{iv}$, $i=1,\ldots,q$; $v=1,\ldots,s$; $s=2^r$, any matrix $\pi_k(T) \in \Pi_k(T)$ can be represented in the form

$$\pi_k(T) = \mathrm{e}^{\tilde{A}_{qs}t_{qs}} \ldots \mathrm{e}^{\tilde{A}_{q1}t_{q1}} \ldots \mathrm{e}^{\tilde{A}_{1s}t_{1s}} \ldots \mathrm{e}^{\tilde{A}_{11}t_{11}},$$

where

$$\sum_{i=1}^{q}\sum_{v=1}^{s} t_{iv} = T, \quad t_{iv} \in R^+, \; i=1,\ldots,q; \; v=1,\ldots,s.$$

Furthermore, there exist constants $L \geqslant 1$ and $\xi > 0$ such that

$$\max_{1\leqslant i\leqslant q, 1\leqslant v\leqslant s} \|\mathrm{e}^{\tilde{A}_{iv}t}\| \leqslant L\mathrm{e}^{-\xi t}, \quad t \in R^+.$$

Therefore, we have the inequality

$$\|\pi_k(T)\| \leqslant L^{qs}\mathrm{e}^{-\xi T}, \quad T \in R^+, \; k \in N. \tag{24}$$

By choosing the real number $\bar{T} > (qs \ln L)/\xi$, we obtain from (24) the inequality (15), which completes the proof of Corollary 4.3. □

Note that sufficiency in Corollary 4.3 can also be proved by using the Lyapunov Second Method (Hahn, 1967; Khalil, 1996). As shown in Narendra and Balakrishnan (1994), in this case, the linear time-varying system (4) has a common quadratic Lyapunov function for any time-varying matrix $A(t) \in \tilde{\mathscr{A}}$.

To formulate the next corollary, we recall that a real matrix $A \in R^{n\times n}$ is said to be normal if $A'A = AA'$ (Axelsson, 1994; Horn & Johnson, 1985; Michel & Herget, 1993). If $A \in R^{n\times n}$ is normal, then $\mu_2(A) = \chi(A)$, where $\mu_2(A)$ is the measure of the matrix A induced by the spectral norm $\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2$. Therefore, using Corollaries 4.1 and 4.2 we obtain the following.

Corollary 4.4. *If every vertex matrix $\tilde{A}_{iv}$, $i = 1,\dots,q$; $v = 1,\dots,s$; $s = 2^r$, defined by* (9) *is normal, then for robust absolute stability of* (1) *with respect to the set $\mathscr{A} \times \Phi$, it is necessary and sufficient that the matrices $\tilde{A}_{iv}$, $i = 1,\dots,q$; $v = 1,\dots,s$; $s = 2^r$, are Hurwitz stable.*

Note that the conditions of Corollary 4.4 ensure the existence of the common Lyapunov function $V(x) = (\|x\|_2)^2 = x'x$ for the system (4) with any time-varying matrix $A(t) \in \tilde{\mathscr{A}}$.

Unfortunately, the condition (15) of Theorem 4.1 is very hard for practical verification even with the use of computers. That is because, in general, we need to check an infinite number of matrices. The situation could be improved if it is possible to establish that the maximum in (23) is attained on a matrix $\tilde{X}_A(T) \in \Omega_0(T)$, and moreover, that we obtain an upper bound for the number of switching points. For the case of $n = 2$ and 3 (second-order and third-order systems), the results concerning the evaluation of the number of switching points have been obtained in Pyatnitskiy and Rapoport (1996) and Rapoport (1994). For systems of higher order ($n \geqslant 4$), the problem of such an evaluation remains open. For this reason it is very important to obtain other necessary and sufficient conditions of robust absolute stability which could be less involved in actual implementations.

By using the Lyapunov Second Method (Hahn, 1967; Khalil, 1996) and its applications in Molchanov and Pyatnitskiy (1986) to the problem of absolute stability of nonlinear continuous-time systems with fixed time-invariant linear part, one can establish necessary and sufficient conditions of robust absolute stability of the system (1) in the form similar to corresponding theorems in Molchanov and Pyatnitskiy (1986). We note that the present results are mostly concerned with identifying the parametric classes of Lyapunov functions defining the necessary and sufficient conditions of robust absolute stability of the system (1).

The main result of the present paper obtained in the framework of this approach is presented in the following theorem.

Theorem 4.2. *For the robust absolute stability of the system* (1) *with respect to the set $\mathscr{A} \times \Phi$, it is necessary and sufficient that for some integer $m \geqslant n$ there exists a full column rank matrix $H \in R^{m\times n}$ and a constant $\theta > 0$ such that a piecewise linear Lyapunov function of the polyhedral vector norm type*

$$V_H(x) = \|x\|_H = \|Hx\|_\infty \tag{25}$$

satisfies the inequality

$$\max_{y \in F(x)} D^+ V_H(x; y) \leqslant -\theta V_H(x), \quad x \in R^n, \tag{26}$$

where $D^+ V_H(x; y) = \lim_{\varepsilon\to 0^+} \varepsilon^{-1}[V_H(x + \varepsilon y) - V_H(x)]$ is the directional derivative of $V_H(x)$ at $x \in R^n$ in the direction $y \in F(x)$ and the set $F(x) \subset R^n$ is defined by (8).

The proof of Theorem 4.2 follows similar steps as in the proof of Theorem 3 given in Part I of Molchanov and Pyatnitskiy (1986) for absolute stability with some nonessential modifications, and therefore, we present here only a general sketch of the proof due to space limitations.

Sufficiency of the theorem is established by usual steps as in the proof of the Lyapunov type theorems on exponential stability (Khalil, 1996; Michel & Miller, 1977). We note that the positive definiteness of the polyhedral Lyapunov function $V_H(x)$ in (25) follows directly from the rank condition

$$\operatorname{rank} H = n \leqslant m \tag{27}$$

on the matrix H. Note also that the inequality (26) guarantees the exponential decreasing of the Lyapunov function $V_H(x)$ along the solutions of the system (1), and the constant θ characterizes the rate of decrease.

To prove necessity, we first construct a strictly convex Lyapunov function $V(x)$ of the vector norm type (in general, nonpolyhedral). In particular, the function $V(x)$ at each point $x_0 \in R^n$ can be defined by the integral $V(x_0) = \int_0^\infty G(T, x_0)\, dT$, where the function $G(T, x_0)$ is defined by (16) with $\|x\| = \|x\|_2$. From the estimate (12) and relation (18) it follows that the function $V(x)$ satisfies the inequality similar to (26) with some constant $\theta_1 > 0$. Then, using the approximation (with arbitrary accuracy) of the level surfaces of the function $V(x)$ by centrally symmetric convex polyhedron (Rockafellar, 1970) we can construct a piecewise linear Lyapunov function $V_H(x)$ of the form (25) which satisfies both the rank condition (27) and the inequality (26). This completes the proof of Theorem 4.2.

From Theorem 4.2 and Corollary 4.1, the following corollary can be obtained.

Corollary 4.5. *The system* (1) *is robustly absolutely stable with respect to the set $\mathscr{A} \times \Phi$, if and only if there exists in R^n a polyhedral vector norm $\|x\|_H$ of the type* (25), *such that the corresponding matrix measure satisfies the condition*

$$\mu_H(\tilde{A}_{iv}) < 0, \quad i = 1,\dots,q; \quad v = 1,\dots,s; \ s = 2^r. \tag{28}$$

Theorem 4.2 makes it also possible to consider Lyapunov functions of the type of even degree $2p$ homogeneous polynomial (Blanchini & Miani, 1999; Molchanov & Pyatnitskiy, 1986, 1989)

$$V_{H,p}(x) = \sum_{i=1}^{m} (H_i x)^{2p} = (\|Hx\|_{2p})^{2p}, \tag{29}$$

where H_i, $i = 1,\dots,m$, is the ith row of matrix H in (25). Under the rank condition (27), the function $V_{H,p}(x)$ will be a strictly convex differentiable function in R^n.

We have the following corollary.

Corollary 4.6. *For the robust absolute stability of the system* (1) *with respect to the set* $\mathscr{A} \times \Phi$, *it is necessary and sufficient that there exists a full column rank matrix* $H \in R^{m\times n}$ (*with* H_i *being its* i*th row*, $i=1,\dots,m$), *a finite integer* $p \geqslant 1$ *and a constant* $\theta > 0$, *such that the Lyapunov function* $V_{H,p}(x)$ *defined by* (29) *satisfies*

$$\max_{y\in F(x)} D^+ V_{H,p}(x;y) = \max_{y\in F(x)} \left\{ 2p \sum_{i=1}^{m} (H_i x)^{2p-1} (H_i y) \right\} \leqslant -\theta V_{H,p}(x), \quad x \in R^n.$$

The proof of Corollary 4.6 is similar to the proof of Theorem 1 in Part II of Molchanov and Pyatnitskiy (1986) and is omitted here.

One of the main advantages of the Lyapunov function $V_{H,p}(x)$ of the form (29) in comparison with the function $V_H(x)$ of the form (25) is that it is a smooth function everywhere in R^n. It gives the opportunity of using a well developed technique of smooth optimization for numerical construction of such functions.

The use of Theorem 4.2 enables us to obtain a criterion for robust absolute stability of the system (1) in algebraic form. Such algebraic criterion of robust absolute stability is stated in the next theorem.

Theorem 4.3. *For the robust absolute stability of the system* (1) *with respect to the set* $\mathscr{A} \times \Phi$, *it is necessary and sufficient that there exists a finite integer* $m \geqslant n$, *a full column rank matrix* $H \in R^{m\times n}$, *and* $m \times m$ *matrices* Γ_{iv}, $i=1,\dots,q$; $v=1,\dots,s$; $s=2^r$, *satisfying*

$$\mu_\infty(\Gamma_{iv}) < 0, \quad i=1,\dots,q; \quad v=1,\dots,s; \; s=2^r, \tag{30}$$

such that the matrix relations

$$H\tilde{A}_{iv} = \Gamma_{iv} H, \quad i=1,\dots,q; \quad v=1,\dots,s; \; s=2^r \tag{31}$$

are satisfied.

The proof of Theorem 4.3 follows with some non-essential modifications from the proof of an analogous criterion in Part III of Molchanov and Pyatnitskiy (1986) (Theorem 1) establishing the necessary and sufficient conditions of absolute stability. Therefore, we provide here a shortened version of the proof.

Necessity: Using the statement of Theorem 4.2, let us denote $W_H(x) = \max_{y\in F(x)} D^+ V_H(x;y)$. In view of (8), (25) and (27), we have $W_H(0) = 0$ and

$$W_H(x) = \max_{1\leqslant i\leqslant q, 1\leqslant v\leqslant s} \max_{j\in J(x)} \{(H_j \tilde{A}_{iv} x)\, \mathrm{sign}(H_j x)\} \tag{32}$$

for $x \neq 0$, where $J(x) = \{j\colon |H_j x| = V_H(x)\} \subset \{1,\dots,m\}$ and H_j, $j=1,\dots,m$, is the jth row of matrix H in (25).

From (26) and (32) it follows that for every $j = 1,\dots,m$; $i = 1,\dots,q$; $v = 1,\dots,s$; $s = 2^r$ the inequality $H_j \tilde{A}_{iv}(\theta) x \leqslant 0$ holds in the domain $K_j^+ = \{x \in R^n\colon H_j x \geqslant |H_\mu x|,\ \mu = 1,\dots,m\}$, where $\tilde{A}_{iv}(\theta) = \tilde{A}_{iv} + \theta I_n$, $i=1,\dots,q$; $v=1,\dots,s$; $s=2^r$.

Applying the Minkowski–Farkas lemma (Rockafellar, 1970) we get equalities $H_j \tilde{A}_{iv}(\theta) = \sum_{\mu=1}^m z_{j\mu}^{(iv)} H_\mu$, or, equivalently,

$$H\tilde{A}_{iv}(\theta) = Z_{iv} H, \tag{33}$$

where each of the $m \times m$ matrices $Z_{iv} = (z_{j\mu}^{(iv)})_{j,\mu=1}^m$ satisfies the condition $\mu_\infty(Z_{iv}) \leqslant 0$, $i=1,\dots,q$; $v=1,\dots,s$; $s=2^r$. The matrix relations (33) are equivalent to (31), where $\Gamma_{iv} = Z_{iv} - \theta I_m$ and $\mu_\infty(\Gamma_{iv}) \leqslant -\theta < 0$, $i=1,\dots,q$; $v=1,\dots,s$; $s=2^r$. Therefore, conditions (30) are valid, and this proves necessity.

Sufficiency: The matrix relations (31) yield for the function $V_H(x)$ of the form (25) the estimates

$$\begin{aligned} D^+ V_H(x;\tilde{A}_{iv}x) &= \lim_{\varepsilon\to 0^+} \varepsilon^{-1}\{\|Hx + \varepsilon H\tilde{A}_{iv}x\|_\infty - \|Hx\|_\infty\} \\ &= \lim_{\varepsilon\to 0^+} \{\|(I_m + \varepsilon\Gamma_{iv})Hx\|_\infty - \|Hx\|_\infty\} \\ &\leqslant \mu_\infty(\Gamma_{iv}) V_H(x) \end{aligned}$$

for every $i=1,\dots,q$; $v=1,\dots,s$; $s=2^r$. Since

$$\max_{y\in F(x)} D^+ V_H(x;y) = \max_{1\leqslant i\leqslant q, 1\leqslant v\leqslant s} D^+ V_H(x;\tilde{A}_{iv}x),$$

we obtain the inequality

$$\max_{y\in F(x)} D^+ V_H(x;y) \leqslant \max_{1\leqslant i\leqslant q, 1\leqslant v\leqslant s} \{\mu_\infty(\Gamma_{iv})\} V_H(x).$$

By (30) the last inequality is reduced to inequality (26), if we choose

$$\theta = \min_{1\leqslant i\leqslant q, 1\leqslant v\leqslant s} \{|\mu_\infty(\Gamma_{iv})|\}. \tag{34}$$

This completes the proof of Theorem 4.3. □

Note that the conditions of Theorem 4.3 are automatically satisfied if $\mu_\infty(\tilde{A}_{iv}) < 0$ for any matrix $\tilde{A}_{iv}$, $i=1,\dots,q$; $v=1,\dots,s$; $s=2^r$. In this case, $H=I_n$, $\Gamma_{iv}=\tilde{A}_{iv}$, $i=1,\dots,q$; $v=1,\dots,s$; $s=2^r$, and $V_H(x) = \|x\|_\infty$ (see also Corollaries 4.1 and 4.5). The inequality $\mu_\infty(A) < 0$ means that the matrix $A \in R^{n\times n}$ is negatively diagonal dominance in rows (Michel & Miller, 1977), since in this case,

$$a_{ii} + \sum_{j=1, j\neq i}^{n} |a_{ij}| < 0 \quad \text{for all } i = 1,2,\dots,n.$$

In connection with Theorem 4.3, we also note that, in fact, the matrix relations (31) are equivalent to the conditions (28) for the matrix measure μ_H induced by the Lyapunov function $V_H(x)$ in (25), which defines a polyhedral vector norm $\|x\|_H$. Moreover, we can choose θ in (26) in accordance with (34). Thus, Theorem 4.2 reduces the problem of robust absolute stability for the system (1) with respect to the set $\mathscr{A} \times \Phi$ to the problem of the existence of a polyhedral vector norm type Lyapunov function (25). In turn, Theorem 4.3 reduces the problem of the construction of such a Lyapunov function to the problem of solvability of the matrix relations (31), which can be considered as a set of matrix equations in terms of unknown matrices $H \in R^{m\times n}$ and $\Gamma_{iv} \in R^{m\times m}$, $i = 1,\dots,q;\ v = 1,\dots,s;\ s = 2^r$.

In general, the problem of solving matrix equations (31), and consequently, the problem of robust absolute stability for the system (1) can effectively be solved only by numerical methods, i.e., with the use of computers. Unfortunately, Theorems 4.2 and 4.3 say nothing about how large an integer m should be to satisfy the conditions of these theorems. In principle, the number m can be much larger than its lower bound n, and it should be chosen experimentally in practical numerical calculations by incrementing its value. If the value of m is fixed (for example, $m = n$), then the conditions of Theorems 4.2 and 4.3 become only sufficient for robust absolute stability of the system (1). We note that an efficient numerical algorithm was proposed in Polański (1997) for solving matrix equations similar to (31) with the use of linear programming and the idea of scaling; the algorithm of Polański (1997) can also be used for checking the conditions of Theorems 4.2 and 4.3. The improvement of these sufficient conditions can be achieved only at the expense of increasing the integer parameter $m > n$.

Nevertheless, for the system (1), simpler sufficient conditions of robust absolute stability, which often coincide with necessary and sufficient conditions, exist in special cases. Some of these cases are discussed next.

5. Sufficient conditions for robust absolute stability

In this section, we establish several simple sufficient conditions for robust absolute stability of the system (1) with respect to the set $\mathscr{A} \times \Phi$. The proposed conditions are special corollaries of the results established in Section 4, and are relatively simple to implement, because these conditions are reduced to checking the Hurwitz stability of some particular test matrix. It is shown that these conditions become necessary and sufficient for several special cases of the system (1).

Let $|A|$ denote the matrix obtained from A by taking the absolute value of all entries. For two $m \times n$ matrices, $A = [a_{ij}] \in R^{m\times n}$ and $B = [b_{ij}] \in R^{m\times n}$, $A \leqslant B$ ($A < B$) denotes an element-wise inequality, i.e., $a_{ij} \leqslant b_{ij}$ ($a_{ij} < b_{ij}$), $i = 1,\dots,m;\ j = 1,\dots,n$. In the inequality $A \geqslant 0$, we use 0 to denote the matrix of appropriate dimension whose entries are all equal to zero. Following Sezer and Šiljak (1994), we will represent a matrix $A = [a_{ij}] \in R^{n\times n}$ as $A = A^{(d)} + A^{(\text{off})}$, where $A^{(d)} = \text{diag}[a_{11}, a_{22}, \dots, a_{nn}]$ and $A^{(\text{off})} = A - A^{(d)}$. Following Luo et al. (1994), we let $A^{\#} = A^{(d)} + |A^{(\text{off})}|$. Obviously, if $A^{\#} \leqslant B^{\#}$, then $\mu_\infty(A) \leqslant \mu_\infty(B)$. Note also that $A^{\#} = A$ holds if and only if $A^{(\text{off})} \geqslant 0$.

Let us associate with a given set of matrices $\tilde{A}_{iv}$, $i = 1,\dots,q;\ v = 1,\dots,s;\ s = 2^r$, defined by (9), a majorant matrix

$$\hat{A} = \max_{1\leqslant i\leqslant q,\ 1\leqslant v\leqslant s} \{\tilde{A}_{iv}^{\#}\}, \tag{35}$$

where the maximum is understood to be element-wise.

The following theorem can now be established.

Theorem 5.1. *The system* (1) *is robustly absolutely stable with respect to the set* $\mathscr{A} \times \Phi$, *if the matrix* $\hat{A}$ *in* (35) *is Hurwitz stable, i.e.,* $\chi(\hat{A}) < 0$.

Proof. Since $\hat{A}^{(\text{off})} \geqslant 0$ and $\chi(\hat{A}) < 0$, then from Theorem 8 in Vidyasagar (1978) it follows that there exists a positive definite diagonal matrix $H \in R^{n\times n}$,

$$H = \text{diag}\{h_1,\dots,h_n\},\ h_i > 0, \quad i = 1,\dots,n \tag{36}$$

such that $\mu_\infty(H\hat{A}H^{-1}) < 0$. Clearly, $\text{rank}(H) = n$. Let us denote $\Gamma = H\hat{A}H^{-1}$ and we have $\mu_\infty(\Gamma) < 0$. Obviously, $H\hat{A} = \Gamma H$ and $\Gamma^{\#} = \Gamma$.

If we now put $\Gamma_{iv} = H\tilde{A}_{iv}H^{-1}$, $i = 1,\dots,q;\ v = 1,\dots,s;\ s = 2^r$, we get the matrix relation (31) with diagonal matrix H defined by (36). Moreover, we have $\Gamma_{iv}^{\#} = H\tilde{A}_{iv}^{\#}H^{-1}$, $\tilde{A}_{iv}^{\#} \leqslant \hat{A}$, $\Gamma_{iv}^{\#} \leqslant \Gamma$, and as a consequence, $\mu_\infty(\Gamma_{iv}) \leqslant \mu_\infty(\Gamma) < 0$, $i = 1,\dots,q;\ v = 1,\dots,s;\ s = 2^r$. According to Theorem 4.3, the system (1) is robustly absolutely stable with respect to the set $\mathscr{A} \times \Phi$, and this completes the proof of Theorem 5.1. □

Note that in the proof of Theorem 5.1, we in essence use the existence for the uncertain system (1) of a linear time-invariant comparison system (Michel & Miller, 1977) $\dot{x} = \hat{A}x$ and the Lyapunov function $V_h(x) = \max_{1\leqslant i\leqslant n}\{h_i|x_i|\}$, $h_i > 0$, $i = 1,\dots,n$, which is a weighted infinity vector norm with weights h_i, $i = 1,\dots,n$. In this case, we can let $\theta = |\mu_\infty(\Gamma)| = |\mu_\infty(H\hat{A}H^{-1})|$ in the corresponding inequality (26).

Theorem 5.1 is a natural generalization of Theorem 1 in Daoyi (1985) for the problem of robust stability of linear time-invariant interval systems. The condition of Theorem 5.1 is only sufficient for robust absolute stability of the system (1). However, if this theorem is applied to systems (1) satisfying some additional conditions, Theorem 5.1 becomes a necessary and sufficient condition for special cases of system (1).

We recall that from Sezer and Šiljak (1994), a matrix $A \in R^{n\times n}$ is called a Morishima matrix if by symmetric row and column permutations it can be transformed into

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{11} \geqslant 0$, $A_{22} \geqslant 0$ are square submatrices and $A_{12} \leqslant 0$, $A_{21} \leqslant 0$. A matrix $A \in R^{n\times n}$ is a Morishima matrix if and only if $SAS = |A|$ for some matrix $S = \mathrm{diag}\{s_1,\ldots,s_n\}$ with $s_i = \pm 1$, $i = 1,\ldots,n$. If $A^{(\mathrm{off})}$ is a Morishima matrix, then $A = A^{(d)} + A^{(\mathrm{off})}$ is Hurwitz stable if and only if $A^{\#} = A^{(d)} + |A^{(\mathrm{off})}|$ is Hurwitz stable (Sezer & Šiljak, 1994).

Our next result characterizes several classes of systems (1) whose robust absolute stability is equivalent to Hurwitz stability of a single test matrix $\hat{A}$ in (35).

Theorem 5.2. *The sufficient condition in Theorem* 5.1 *is also necessary for each of the following cases of system* (1).

(i) *In the given set of vertex matrices* $\tilde{A}_{iv}$, $i=1,\ldots,q$; $v=1,\ldots,s$; $s=2^r$, *defined by* (9), *there exists at least one matrix* $\tilde{A}_{\hat{i}\hat{v}}$ *such that* $\tilde{A}_{\hat{i}\hat{v}} = \hat{A}$.

(ii) *In the given set of vertex matrices* $\tilde{A}_{iv}$, $i=1,\ldots,q$; $v=1,\ldots,s$; $s=2^r$, *defined by* (9), *there exists at least one matrix* $\tilde{A}_{\hat{i}\hat{v}}$ *such that* $\tilde{A}^{\#}_{\hat{i}\hat{v}} = \hat{A}$ *holds and* $\tilde{A}^{(\mathrm{off})}_{\hat{i}\hat{v}}$ *is a Morishima matrix.*

(iii) *All matrices* $\tilde{A}_{iv}$, $i=1,\ldots,q$; $v=1,\ldots,s$; $s=2^r$, *defined by* (9), *are either all upper triangular or all lower triangular.*

Proof. In case (i), the proof follows immediately from Corollary 4.2 which implies $\chi(\hat{A}) = \chi(\tilde{A}_{\hat{i}\hat{v}}) < 0$.

In case (ii), from Corollary 4.2, it follows that the matrix $\tilde{A}_{\hat{i}\hat{v}}$ is Hurwitz stable. Since $\tilde{A}^{(\mathrm{off})}_{\hat{i}\hat{v}}$ is a Morishima matrix, then the matrix $\tilde{A}^{\#}_{\hat{i}\hat{v}} = \hat{A}$ is also Hurwitz stable.

In case (iii), if all matrices $\tilde{A}_{iv}$, $i=1,\ldots,q$; $v=1,\ldots,s$; $s=2^r$, are upper triangular, then from the definition in (35) for the matrix $\hat{A}$, it follows that it is also upper triangular and

$$\chi(\hat{A}) = \max_{1\leqslant i\leqslant q, 1\leqslant v\leqslant s} \{\chi(\tilde{A}_{iv})\}.$$

From Corollary 4.2, we have $\chi(\tilde{A}_{iv}) < 0$, $i=1,\ldots,q$; $v=1,\ldots,s$; $s=2^r$, and consequently, $\chi(\hat{A}) < 0$, i.e., the matrix $\hat{A}$ is Hurwitz stable. In the case of lower triangular matrices $\tilde{A}_{iv}$, $i=1,\ldots,q$; $v=1,\ldots,s$; $s=2^r$, the proof is fully analogous. This completes the proof of Theorem 5.2. □

Note that in the case of upper or lower triangular matrices $\tilde{A}_{iv}$, $i=1,\ldots,q$; $v=1,\ldots,s$; $s=2^r$, the robust absolute stability of the system (1) depends only on their diagonal elements. Therefore, the off-diagonal elements have no effect on robust absolute stability of the system (1) in this case. The results of Theorem 5.2 are generalizations of results for linear time-invariant interval systems derived in Sezer and Šiljak (1994).

6. Application examples

In this section, the use of the present results to analyze the robust absolute stability of the system (1) will be illustrated by two examples.

Example 1. Let us consider the second-order system (1) ($n=2$) with $q=2$ and $r=1$, where

$$A_1 = \begin{bmatrix} -3 & -1 \\ -4 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -5 & 1 \\ -2 & -4 \end{bmatrix},$$

$$b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

$\alpha = 0$ and $\beta \geqslant 0$. In this case, $s = 2^r = 2$ and the matrices $\tilde{A}_{iv}$, $i, v = 1, 2$ of the form (9) are given by

$$\tilde{A}_{11} = A_1 = \begin{bmatrix} -3 & -1 \\ -4 & -2 \end{bmatrix},$$

$$\tilde{A}_{12} = A_1 + \beta bc' = \begin{bmatrix} -3+\beta & -1-\beta \\ -4 & -2 \end{bmatrix},$$

$$\tilde{A}_{21} = A_2 = \begin{bmatrix} -5 & 1 \\ -2 & -4 \end{bmatrix},$$

$$\tilde{A}_{22} = A_2 + \beta bc' = \begin{bmatrix} -5+\beta & 1-\beta \\ -2 & -4 \end{bmatrix}.$$

It can be verified that

$$\hat{A} = \tilde{A}^{\#}_{12} = \begin{bmatrix} -3+\beta & 1+\beta \\ 4 & -2 \end{bmatrix}.$$

Since the matrix

$$\tilde{A}^{(\mathrm{off})}_{12} = \begin{bmatrix} 0 & -1-\beta \\ -4 & 0 \end{bmatrix}$$

is a Morishima matrix for any $\beta \geqslant 0$, then according to Theorem 5.1 and the point (ii) of Theorem 5.2, the condition $\chi(\hat{A}) < 0$, i.e., the condition $0 \leqslant \beta < \frac{1}{3}$, is necessary and sufficient for robust absolute stability of the system under consideration.

Example 2. Let us consider the case when the set $\mathscr{A}$ defined by (2) is a family of time-varying interval matrices given by $\mathscr{A}(\underline{A}, \bar{A}) = \{A(t) \in R^{n\times n}: \underline{A} \leqslant A(t) \leqslant \bar{A}$, for almost all $t \in R^+\}$, where $\underline{A} = [\underline{a}_{ij}] \in R^{n\times n}$ and $\bar{A} = [\bar{a}_{ij}] \in R^{n\times n}$ are fixed matrices, and the inequalities are element-wise.

When $b_j c_j' \geqslant 0$, $j=1,\ldots,r$, the set $\tilde{\mathscr{A}}$ defined by (10) is contained in the family of time-varying interval matrices given by $\tilde{\mathscr{A}}(\underline{A}_l, \bar{A}_u) = \{A(t) \in R^{n\times n}: \underline{A}_l \leqslant A(t) \leqslant \bar{A}_u$, for almost all $t \in R^+\}$, where $\underline{A}_l = \underline{A} + \sum_{j=1}^{r} \alpha_j b_j c_j'$ and $\bar{A}_u = \bar{A} + \sum_{j=1}^{r} \beta_j b_j c_j'$. We note that in this case the conditions $\chi(\underline{A}_l) < 0$ and $\chi(\bar{A}_u) < 0$ are necessary for robust stability of the linear system (4) with respect to both of the sets $\tilde{\mathscr{A}}$ and $\tilde{\mathscr{A}}(\underline{A}_l, \bar{A}_u)$. A corresponding matrix $\hat{A}$ defined by (35) is determined as

$$\hat{A} = \max\{\underline{A}_l^{\#}, \bar{A}_u^{\#}\} = \bar{A}_u^{(d)} + \max\{|\underline{A}_l^{(\mathrm{off})}|, |\bar{A}_u^{(\mathrm{off})}|\},$$

where the maximum is element-wise. If $|\underline{A}_l^{(\text{off})}| \leqslant \bar{A}_u^{(\text{off})}$, then $\hat{A} = \bar{A}_u = \bar{A} + \sum_{j=1}^r \beta_j b_j c_j'$. Therefore, the condition of Theorem 5.2(i) is fulfilled, and the necessary and sufficient condition for robust absolute stability of the system (1) is the inequality $\chi(\bar{A} + \sum_{j=1}^r \beta_j b_j c_j') < 0$, i.e., the matrix $\bar{A}_u = \bar{A} + \sum_{j=1}^r \beta_j b_j c_j'$ is Hurwitz stable. If $|\bar{A}_u^{(\text{off})}| \geqslant |\underline{A}_l^{(\text{off})}|$, then $\hat{A} = \bar{A}_u^{\#}$ and, under the additional assumption that $\bar{A}_u^{(\text{off})}$ is a Morishima matrix, the matrices $\bar{A}_u$ and $\bar{A}_u^{\#}$ will be Hurwitz stable at the same time. In this case, the condition of Theorem 5.2(ii) is fulfilled and we can conclude the same as above.

We can obtain similar results for the case when $b_j c_j' \leqslant 0$, $j = 1, \ldots, r$. In this case, we only need to replace α_j in the above formulas for $\underline{A}_l$ and $\bar{A}_u$ by β_j, $j = 1, \ldots, r$, and vice versa.

7. Conclusions

In this paper, we have investigated the problem of robust absolute stability of a class of nonlinear continuous-time systems with time-varying matrix uncertainty of polyhedral type and with multiple time-varying sector nonlinearities. By using the variational method and the Lyapunov Second Method, necessary and sufficient conditions for robust absolute stability have been obtained in different forms for the given class of systems. It was shown that in general, the problem of checking these conditions can be effectively solved by numerical methods, admitting a computer-aided implementation. Several simple sufficient conditions for robust absolute stability have been provided which become necessary and sufficient for special cases. As examples, we have applied the present results to a particular second-order system and to a specific class of systems with time-varying interval matrices in the linear part.

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