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Asymptotic stability of systems operating on a closed hypercube *

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Abstract: Sufficient conditions for the global asymptotic stability of the equilibrium $x_e = 0$ of dynamical systems which are characterized by linear ordinary differential equations with saturation nonlinearities are established. The class of systems considered herein arises in the modeling of control systems and neural networks.

Keywords: Asymptotic stability; nonlinear systems; continuous-time systems; state saturation.

1. Introduction

In this paper, we will investigate stability properties of systems described by

$$\dot{x}(t) = h[Ax(t)], \quad t \geqslant 0, \tag{1}$$

where $x(t) \in D^n := \{x \in R^n: -1 \leqslant x_i \leqslant 1, i = 1,\dots,n\}$, $A = [a_{ij}] \in R^{n \times n}$,

$$h(Ax) = \left[h\left(\sum_{j=1}^{n} a_{1j}x_j\right),\dots,h\left(\sum_{j=1}^{n} a_{nj}x_j\right)\right]^{\mathrm{T}},$$

and

$$h\left(\sum_{j=1}^{n} a_{ij}x_j\right) = \begin{cases} 0, & x_i = 1, \\ \sum_{j=1}^{n} a_{ij}x_j, & -1 < x_i < 1, \quad \text{for } i = 1,\dots,n. \\ 0, & x_i = -1 \end{cases}$$

We will refer to (1) as a *'linear' system operating on a closed hypercube.*

Equation (1) represents a class of continuous-time dynamical systems with symmetrically saturating states after normalization. Examples of such systems include control systems (see [3] for a discussion of discrete-time systems of this type) and certain classes of neural networks [1,2].

When considering (1) as a *control system* (with no external inputs), some of the first fundamental questions that arise concern the existence and uniqueness of an equilibrium or operating point (which we assume to be the origin, without loss of generality) and the qualitative properties (specifically, stability properties) of such an equilibrium. The condition that the matrix A be stable (i.e., that all of the eigenvalues of A be located in the left half complex plane) does not ensure that $x_e = 0$ is asymptotically stable in the large. For example, the matrix

$$A = \begin{pmatrix} 11.1 & -20 & 4 & -7 \\ 30 & -30 & -1 & -19.5 \\ 8.4 & 6.6 & 10 & -20 \\ 10 & -10 & 30 & -30 \end{pmatrix}$$

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has eigenvalues $\lambda(A) = -0.2921, -28.5009, -5.0535 \pm 21.6362i$, i.e., A is stable. It is easily verified that in addition to the origin, the system (1) with A specified above, has also equilibria at

$$x_1 = (-1, -0.3167, -1, -1)^T \quad \text{and} \quad x_2 = (1, 0.3167, 1, 1)^T.$$

Thus, while $x_e = 0$ is certainly asymptotically stable, it is not asymptotically stable in the large.

When considering system (1) as a *neural network with applications to optimization problems*, we wish to construct a network with a unique equilibrium which is globally asymptotically stable, in order to prevent convergence to local minima of an objective function (see, e.g., [4]).

In the present paper we will establish a set of sufficient conditions which ensure the global asymptotic stability of the equilibrium $x_e = 0$ of system (1).

The remainder of this paper is organized as follows. In Section 2, we introduce some essential notation, and in Section 3, we present our main result. We conclude the present paper in Section 4.

2. Notation

For $x \in R^n$, we define the l_p vector norm as

$$\| x \|_p = \left(\sum_{i=1}^{n} | x_i |^p \right)^{1/p}, \quad \text{for } 1 \leqslant p \leqslant \infty.$$

Recall that when $p = \infty$, we have $\| x \|_\infty = \max_{1 \leqslant i \leqslant n}\{ | x_i | \}$. For $A \in R^{n \times n}$, we define the norm of A by

$$\| A \| = \inf\{ \gamma : \| Ax \| \leqslant \gamma \| x \| \text{ for all } x \in R^n \}.$$

Recall that for $p = \infty$, the norm of A, induced by the l_∞ vector norm, is given by $\| A \|_\infty = \max_{1 \leqslant i \leqslant n}\{ \Sigma_{j=1}^{n} | a_{ij} | \}$.

The measure of a matrix $A \in R^{n \times n}$ is defined as

$$\mu_p(A) = \lim_{\theta \to 0^+} \frac{\| I + \theta A \|_p - 1}{\theta},$$

where $\| \cdot \|_p$ denotes the matrix norm induced by the l_p vector norm and I is the identity matrix. In particular, when $p = \infty$, we have $\mu_\infty(A) = \max_{1 \leqslant i \leqslant n}\{ a_{ii} + \Sigma_{j=1, j \neq i}^{n} | a_{ij} | \}$.

We denote the interior and the boundary of a set Ω by $(\Omega)^\circ$ and $\partial\Omega$, respectively.

For $x = (x_1, \ldots, x_n)^T$ and $y = (y_1, \ldots, y_n)^T$, we let $x * y = (x_1 y_1, \ldots, x_n y_n)^T$, and we let $\min(x) = \min_{1 \leqslant i \leqslant n}\{x_i\}$. Also, the notation $x \leqslant y$ will mean $x_i \leqslant y_i$ for $1 \leqslant i \leqslant n$.

3. Main result

We recall that for a general autonomous system

$$\dot{x} = f(x), \tag{2}$$

with $x \in R^n$ and $f : R^n \to R^n$, x_e is an equilibrium for (2) if and only if

$$f(x_e) = 0.$$

We can assume, without loss of generality, that $x_e = 0$ (see, e.g., [5]). Thus, for system (1), we assume that A is nonsingular.

We are now in a position to establish the following result.

Theorem. *The equilibrium* $x_e = 0$ *of system* (1) *is globally asymptotically stable, if*

$$\mu_\infty(A) < 0. \tag{3}$$

Since $\operatorname{Re} \lambda(A) \leqslant \mu_\infty(A) < 0$ (see [6]), the equilibrium $x_e = 0$ is clearly asymptotically stable. We need only to prove that it is also *globally* asymptotically stable. We will prove this in the following two steps.

(1) For any $x(0) \in D^n$, $x(t)$ will not always stay on ∂D^n for $t > 0$.

(2) $x(t) \to 0$ as $t \to \infty$ for any $x(0) \in (D^n)^0$.

In the proof of he first step, we will utilize the notation given below, which was first introduced in [1] and [2].

For each integer m, $0 \leqslant m \leqslant n$, let

$$\Lambda_m = \left\{\xi = (\xi_1, \dots, \xi_n)^{\mathrm{T}} \in \Lambda \colon \xi_{\sigma(i)} = 0,\ 1 \leqslant i \leqslant m \text{ and } \xi_{\sigma(i)} = \pm 1,\ m < i \leqslant n, \text{ for some } \sigma \in \operatorname{Sym}(n)\right\}$$

where $\Lambda = \{\xi = (\xi_1, \dots, \xi_n)^{\mathrm{T}} \colon \xi_i = \pm 1 \text{ or } 0,\ 1 \leqslant i \leqslant n\}$ and let $\operatorname{Sym}(n)$ denote the symmetric group of order n. For each $\xi \in \Lambda$, let

$$C(\xi) = \left\{x = (x_1, \dots, x_n)^{\mathrm{T}} \in R^n \colon |x_i| < 1 \text{ if } \xi_i = 0, \text{ and } x_i = \xi_i \text{ if } \xi_i \neq 0\right\}.$$

Suppose that $\xi \in \Lambda_m$ and $\sigma \in \operatorname{Sym}(n)$ such that

$$\xi_{\sigma(i)} = 0,\ 1 \leqslant i \leqslant m \quad \text{and} \quad \xi_{\sigma(i)} = \pm 1,\ m < i \leqslant n. \tag{4}$$

We denote

$$A_{\mathrm{I,I}} = [a_{\sigma(i)\sigma(j)}]_{1 \leqslant i,j \leqslant m}, \quad A_{\mathrm{I,II}} = [a_{\sigma(i)\sigma(j)}]_{1 \leqslant i \leqslant m, m < j \leqslant n},$$
$$A_{\mathrm{II,I}} = [a_{\sigma(i)\sigma(j)}]_{m < i \leqslant n, 1 \leqslant j \leqslant m}, \quad A_{\mathrm{II,II}} = [a_{\sigma(i)\sigma(j)}]_{m < i,j \leqslant n},$$

and

$$\xi_{\mathrm{I}} = (\xi_{\sigma(1)}, \dots, \xi_{\sigma(m)})^{\mathrm{T}}, \qquad \xi_{\mathrm{II}} = (\xi_{\sigma(m+1)}, \dots, \xi_{\sigma(n)})^{\mathrm{T}}.$$

Remark 1. For a given $\xi \in \Lambda_m$, there may exist different permutations in $\operatorname{Sym}(n)$ for which (4) is true. For these different permutations, the notation given above will be the same up to different orders in the components. Thus, the subsequent analysis and conclusions will be identical for any of the permutations used.

Remark 2. If $m = n$, we have $A_{\mathrm{I,I}} = A$, $\xi_{\mathrm{I}} = \xi$ and the $A_{\mathrm{I,II}}$, $A_{\mathrm{II,I}}$, $A_{\mathrm{II,II}}$, ξ_{II} do not exist. If $m = 0$, we have $A_{\mathrm{II,II}} = A$, $\xi_{\mathrm{II}} = \xi$ and the $A_{\mathrm{I,I}}$, $A_{\mathrm{I,II}}$, $A_{\mathrm{II,I}}$, ξ_{I} do not exist.

Proof of the Theorem. (1) Consider $\xi \in \Lambda_m$, $0 < m < n$, with $\sigma \in \operatorname{Sym}(n)$ such that $\xi_{\sigma(i)} = 0$, $1 \leqslant i \leqslant m$, and $\xi_{\sigma(i)} = \pm 1$, $m < i \leqslant n$. When $x \in C(\xi)$, system (1) becomes

$$\dot{x}_{\mathrm{I}} = A_{\mathrm{I,I}} x_{\mathrm{I}} + A_{\mathrm{I,II}} \xi_{\mathrm{II}}, \qquad \dot{x}_{\mathrm{II}} = 0 \tag{5}$$

where

$$x_{\mathrm{I}} = (x_{\sigma(1)}, \dots, x_{\sigma(m)})^{\mathrm{T}}, \qquad x_{\mathrm{II}} = (x_{\sigma(m+1)}, \dots, x_{\sigma(n)})^{\mathrm{T}} = (\xi_{\sigma(m+1)}, \dots, \xi_{\sigma(n)})^{\mathrm{T}} = \xi_{\mathrm{II}},$$

and

$$-1 < x_{\sigma(i)} < 1 \quad \text{for } 1 \leqslant i \leqslant m.$$

In order to satisfy $\dot{x}_{\mathrm{II}} = 0$, i.e., in order to maintain x in $C(\xi)$, it is necessary that (cf. [1,2])

$$\min((A_{\mathrm{II,I}} x_{\mathrm{I}} + A_{\mathrm{II,II}} \xi_{\mathrm{II}}) * \xi_{\mathrm{II}}) \geqslant 0. \tag{6}$$

Denote $A_{\mathrm{II,I}} = [a_{ij}^{(1)}] \in R^{(n-m)\times m}$, $A_{\mathrm{II,II}} = [a_{ij}^{(2)}] \in R^{(n-m)\times(n-m)}$,

$$x_{\mathrm{I}} = (x_{\sigma(1)},\ldots,x_{\sigma(m)})^{\mathrm{T}} = (x_1^{(1)},\ldots,x_m^{(1)})^{\mathrm{T}}, \quad \text{and} \quad \xi_{\mathrm{II}} = (\xi_{\sigma(m+1)},\ldots,\xi_{\sigma(n)})^{\mathrm{T}} = (\xi_1^{(2)},\ldots,\xi_{n-m}^{(2)})^{\mathrm{T}}.$$

Then, we have

$$\begin{aligned}(A_{\mathrm{II,I}}x_{\mathrm{I}} + A_{\mathrm{II,II}}\xi_{\mathrm{II}}) * \xi_{\mathrm{II}} &= (A_{\mathrm{II,I}}x_{\mathrm{I}}) * \xi_{\mathrm{II}} + (A_{\mathrm{II,II}}\xi_{\mathrm{II}}) * \xi_{\mathrm{II}} \\ &= \left(\xi_1^{(2)} \sum_{j=1}^{m} a_{1j}^{(1)} x_j^{(1)},\ldots,\xi_{n-m}^{(2)} \sum_{j=1}^{m} a_{n-m,j}^{(1)} x_j^{(1)}\right)^{\mathrm{T}} \\ &\quad + \left(\xi_1^{(2)} \sum_{j=1}^{n-m} a_{1j}^{(2)} \xi_j^{(2)},\ldots,\xi_{n-m}^{(2)} \sum_{j=1}^{n-m} a_{n-m,j}^{(2)} \xi_j^{(2)}\right)^{\mathrm{T}}. \end{aligned} \tag{7}$$

By noting that $|x_{\sigma(i)}| < 1$ for $0 \leqslant i \leqslant m$ and $\xi_{\sigma(i)} = \pm 1$ for $m < i \leqslant n$, we have

$$\begin{aligned}&(A_{\mathrm{II,I}}x_{\mathrm{I}} + A_{\mathrm{II,II}}\xi_{\mathrm{II}}) * \xi_{\mathrm{II}} \\ &\leqslant \left(\sum_{j=1}^{m} |a_{1j}^{(1)}|, \sum_{j=1}^{m} |a_{2j}^{(1)}|,\ldots,\sum_{j=1}^{m} |a_{n-m,j}^{(1)}|\right)^{\mathrm{T}} \\ &\quad + \left(a_{11}^{(2)} + \sum_{j=2}^{n-m} |a_{1j}^{(2)}|, a_{22}^{(2)} + \sum_{j=1,j\neq 2}^{n-m} |a_{2j}^{(2)}|,\ldots,a_{n-m,n-m}^{(2)} + \sum_{j=1}^{n-m-1} |a_{n-m,j}^{(2)}|\right)^{\mathrm{T}} \\ &= \left(a_{11}^{(2)} + \sum_{j=1}^{m} |a_{1j}^{(1)}| + \sum_{j=2}^{n-m} |a_{1j}^{(2)}|, a_{22}^{(2)} + \sum_{j=1}^{m} |a_{2j}^{(1)}| + \sum_{j=1,j\neq 2}^{n-m} |a_{2j}^{(2)}|,\ldots,\right. \\ &\qquad \left. a_{n-m,n-m}^{(2)} + \sum_{j=1}^{m} |a_{n-m,j}^{(1)}| + \sum_{j=1}^{n-m-1} |a_{n-m,j}^{(2)}|\right)^{\mathrm{T}}. \end{aligned} \tag{8}$$

Notice that the entries in the right hand side of (8) are just rearrangements of

$$a_{ii} + \sum_{j=1,j\neq i}^{n} |a_{ij}|, \quad \text{for } i = \sigma(m+1),\ldots,\sigma(n),$$

and thus, since $\mu_\infty(A) < 0$ (by assumption), every entry in the right hand side of (8) is less than 0. Therefore, condition (6) will fail to hold and this is true for every m, $0 < m < n$. It is also true for $m = 0$ by noting that $(A\eta) * \eta < 0$ for any $\eta \in \Lambda_0$ when $\mu_\infty(A) < 0$. Thus, the state of system (1) will not stay on the boundary of D^n for all time t, since

$$\partial D^n = \bigcup_{m=0}^{n-1} \{C(\xi)\colon \xi \in \Lambda_m\}.$$

(2) Thus far, we have proved that for any $x(0) \in D^n$, it is impossible for $x(t)$ to remain in ∂D^n for *all* $t > 0$. We now show that under the conditions of our Theorem, once $x(t)$ leaves ∂D^n, it will never enter ∂D^n again.

Therefore, without loss of generality, we assume that $x(0) \in (D^n)^\circ$. Then, system (1) is equivalent to

$$\dot{x} = Ax \tag{9}$$

as long as $x(t)$ does not reach ∂D^n and the solution for (9) is given by

$$x(t) = \mathrm{e}^{At}x(0).$$

We compute $\| x(t)\|_\infty = \|e^{At}x(0)\|_\infty \leqslant \|e^{At}\|_\infty \| x(0)\|_\infty$. Using the fact that (see [6], p.59) $\|e^{At}\|_p \leqslant e^{\mu_p(A)t}$, for any $t \geqslant 0$ and any $p \geqslant 1$, we have

$$\| x(t)\|_\infty \leqslant e^{\mu_\infty(A)t} \| x(0)\|_\infty < \| x(0)\|_\infty, \quad \text{for any } t > 0,$$

since $\mu_\infty(A) < 0$. This in turn implies that

$$x(t) \in (D^n)^\circ, \textit{ for all } t > 0,$$

i.e., when $x(0) \in (D^n)^\circ$, $x(t)$ will *never* reach the boundary of D^n. Therefore, system (1) is equivalent to (9) *for all* $t \geqslant 0$ when $x(0) \in (D^n)^\circ$. Hence, $x(t) \to 0$ as $t \to \infty$, since $\operatorname{Re} \lambda(A) < 0$. □

Summarizing, above we have shown that
(i) $x(0) \notin D^n$ is not allowed;
(ii) if $x(t_1) \in \partial D^n$, $x(t)$ cannot stay in ∂D^n for all $t > t_1$; and
(iii) once $x(t)$ is in $(D^n)^\circ$, it will never enter ∂D^n and $x(t) \to 0$ as $t \to \infty$.

Remark 3. In [3], we consider the discrete-time counterpart of system (1), given by

$$x(k+1) = \operatorname{sat}[Ax(k)], \quad k = 0, 1, \ldots, \tag{10}$$

where $x \in R^n$, $A \in R^{n \times n}$, and

$$\operatorname{sat}(x) = [\operatorname{sat}(x_1), \ldots, \operatorname{sat}(x_n)]^T \quad \text{with} \quad \operatorname{sat}(x_i) = \begin{cases} 1, & x_i \geqslant 1, \\ x_i, & -1 < x_i < 1, \\ -1, & x_i \leqslant -1. \end{cases}$$

The condition which ensures the equilibrium $x_e = 0$ of system (10) to be globally asymptotically stable (see [3]), given by

$$\| A \|_\infty < 1, \tag{11}$$

constitutes a discrete-time counterpart to condition (3).

4. Conclusion

Equation (1) describes a class of continuous-time dynamical systems with state saturation nonlinearities – special kinds of hard limiters. Systems of this type arise frequently in the modeling of control systems and neural networks. Thus, the stability properties of such systems are of great interest. Our result states that the null solution of system (1) will be globally asymptotically stable, if the measure of the coefficient matrix A, induced by the matrix $\| \cdot \|_\infty$ norm is negative. This suggests that matrix measure may play an important role in the stability analysis of systems described by (1).

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