Asymptotic Stability of Discrete-Time Systems with Saturation Nonlinearities with Applications to Digital Filters

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***Abstract*—We establish new results for the global asymptotic stability of the equilibrium $x = 0$ of nth-order discrete-time systems with state saturations, $x(k+1) = \text{sat}[Ax(k)]$ (where $A \in R^{n \times n}$ and sat $(\cdot)$ denotes the normalized, symmetric saturation nonlinearity for each vector component). Associated with such systems are linear systems of the form $w(k+1) = Aw(k)$. In our approach, we utilize a class of positive definite and radially unbounded Lyapunov functions $v(\cdot)$ with the properties that $v(w(k+1)) - v(w(k)) = v(Aw(k)) - v(w(k))$ is negative definite and $v(\text{sat}(w)) < v(w)$ when $w \notin D^n \triangleq \{w \in R^n\text{: } -1 \le w_i \le 1,\ i = 1,\cdots,n\}$. For the case when v is a quadratic form, we establish necessary and sufficient conditions under which positive definite matrices H can be used to generate Lyapunov function $v(w) = w^T Hw$ with the desired properties that $v(Aw(k)) - v(w(k))$ is negative semidefinite, and that $v(\text{sat}(w)) < v(w)$ when $w \notin D^n$. This Lyapunov function $v(\cdot)$ is then used in the stability analysis of systems described by the equation, $x(k+1) = \text{sat}[Ax(k)]$.**

For the nth-order fixed-point digital filters, we review some of the existing results and utilize the above results to establish conditions for the non-existence of limit cycles in such filters. We demonstrate that the present results are easier to apply and are less conservative than corresponding existing results.

I. Introduction

In this paper we will investigate stability properties of systems described by

$$x(k+1) = \text{sat}[Ax(k)], \qquad k = 0, 1, 2, \cdots \tag{1}$$

where $x(k) \in D^n \triangleq \{x \in R^n\text{: } -1 \le x_i \le 1,\ i = 1,\cdots,n\}$, $A \in R^{n \times n}$,

$$\text{sat}(x) = [\text{sat}(x_1), \text{sat}(x_2), \cdots, \text{sat}(x_n)]^T$$

and

$$\text{sat}(x_i) = \begin{cases} 1, & x_i > 1 \\ x_i, & -1 \le x_i \le 1 \\ -1, & x_i < -1. \end{cases}$$

We will say that *system* (1) *is stable* if $x_e = 0$ is the only equilibrium of system (1) and $x_e = 0$ is globally asymptotically stable. (Recall that the equilibrium $x_e = 0$ of system (1) is *globally asymptotically stable* if (i) it is *stable* in the sense of Lyapunov, i.e., for every $\epsilon > 0$ there is a $\delta = \delta(\epsilon)$ such that $\|x(k)\| < \epsilon$ for all $k = 0, 1, 2, \cdots$, whenever $\|x(0)\| < \delta$ ($\|\cdot\|$ denotes any vector norm), and (ii) it is *attractive, i.e.*, $x(k) \to 0$ as $k \to \infty$.) Clearly, if for (1) $x_e = 0$ is globally asymptotically stable, then (1) will not possess any limit cycles. Also, since we have saturation nonlinearities in (1), it is clear that for any $x(0) \notin D^n$, $x(k) \in D^n$, $k \ge 1$, will always be true. Thus without loss of generality, we will assume that $x(0) \in D^n$.

Equation (1) describes a class of discrete-time dynamical systems with symmetrically saturating states after normalization. Examples of such systems include control systems having saturation type nonlinearities on the state (cf. [17], [41]); neural networks defined on hypercubes (cf. [13], [14]); and digital filters using saturation overflow arithmetic (see, e.g., [18]–[26], [29]–[40]). We will not consider non-symmetric state saturation in the present paper.

1) *Control systems* with saturation on the controllers are still under investigation (see e.g., [4]–[12]). In these studies, it is generally assumed that there is no state saturation in the system. In practice, this is not realistic. For example, in describing the dynamics of a car, we may choose speed and steering angle as two of the state variables. Since both of these variables have upper and lower limits, this system is endowed with state saturation nonlinearities. In applications, state saturation in control systems is very common.

System (1) may be used to represent control systems with saturating states with no external inputs. In the analysis and design of such systems, the first fundamental question addresses stability: under what conditions is $x_e = 0$ an equilibrium and when is this equilibrium globally asymptotically stable?

The condition that A is a stable matrix, i.e., every eigenvalue λ_i of A satisfies $|\lambda_i| < 1$, is not sufficient for system (1) to be stable. (It is easy to give examples for which A is a stable matrix, but system (1) is not stable.) One way of guaranteeing the stability of system (1) is to consider D^n as a state constraint set which is positively invariant and contractive [1]–[3] with respect to the linear system

$$x(k+1) = Ax(k) \tag{2}$$

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(i.e., for (2), $x \in D^n$ implies $Ax \in D^n$, and if $x(0) \in D^n$, then $x(k) \to 0$, as $k \to \infty$). This is true *if and only if*

$$\|A\|_\infty < 1 \tag{3}$$

where $\|\cdot\|_\infty$ represents the matrix norm induced by the l_∞ vector norm. Condition (3) will guarantee the global asymptotic stability of the equilibrium $x_e = 0$ of system (1) since under this condition, system (1) and system (2) are equivalent.

Condition (3) may also be viewed as a direct application of the results in [1]–[3], where necessary and sufficient conditions for a polyhedral state constraint set to be positively invariant and contractive are given. We point here the difference between a system with state saturation nonlinearity and a system with state constraint set D^n. The former is a system with a nonlinear property, while the latter is a system whose states are not allowed to violate a constraint set. It is expected that the condition that system (1) is stable should be less conservative than the condition that D^n is a contractive and positively invariant set for the system (2), i.e., the condition (3) is too conservative for the stability of the system (1). We will see in Section II that condition (3) is a special case of the results of the present paper.

2) Systems described by (1) have also been used to represent a class of *neural networks* (cf. [13], [14]). It is shown in [13] that neural networks described by (1) have certain advantages over the Hopfield model. When considering system (1) as a neural network with applications to associative memories, the design objective is to generate a system which stores a set of desired patterns as asymptotically stable equilibrium points. In the application of neural networks to optimization problems (cf. [15]), we wish to construct a network with a unique equilibrium which is globally asymptotically stable, in order to prevent convergence to local minima of an objective function (see, e.g., [16]). When the desired equilibrium x_d is located in the interior of D^n, the conditions for this equilibrium to be globally asymptotically stable will be identical to the conditions for the equilibrium $x_e = 0$ of (1) to be globally asymptotically stable, since we can always consider $x_d = 0$, without loss of generality (cf. [27]).

3) In many important applications, (1) may be used to represent *digital processing systems*, including *digital filters* and digital control systems (cf. [17]–[26], [29]–[41]) with finite wordlength arithmetic under zero external inputs. In such systems, saturation arithmetic is used to cope with the overflow. The absence of limit cycles in such systems is of great interest and can be guaranteed by the global asymptotic stability of the equilibrium $x_e = 0$ for (1). The Lyapunov theory has been found to be an appropriate method for solving such problems (cf. [18]–[21]). We will review further some of these results in Section III.

The remainder of this paper is organized as follows. In Section II, we establish results for the global asymptotic stability of system (1). In Section III, we address applications to digital filters. In Section IV, we consider several specific examples to demonstrate the applicability of the present results. A few pertinent remarks are given in the last section, Section V. Details concerning proofs of some of the results (of Sections II and III) are included in the Appendix.

II. Main Results

In establishing our results, we will make use of Lyapunov functions for the linear systems corresponding to the system (1), given by

$$w(k+1) = Aw(k), \qquad k = 0,1,2,\cdots \tag{4}$$

where $A \in R^{n\times n}$ is defined in (1).

We recall that for a general autonomous system

$$x(k+1) = f(x(k)), \qquad k = 0,1,2,\cdots \tag{5}$$

with $x(k) \in R^n$ and $f: R^n \to R^n$, x_e is an equilibrium for (5) if and only if

$$x_e = f(x_e).$$

We assume, without loss of generality that $x_e = 0$ (see, e.g., [27], [28]). Recall also that the equilibrium $x_e = 0$ for the system (5) is globally asymptotically stable, if there exists a continuous function $v: R^n \to R$ which is positive definite, radially unbounded, and along solutions of (5) satisfies the condition that

$$\begin{aligned} Dv_{(5)}(x(k)) &\triangleq v(x(k+1)) - v(x(k)) \\ &= v(f(x(k))) - v(x(k)) \end{aligned} \tag{6}$$

is negative definite for all $x(k) \in R^n$. The function v is an example of a *Lyapunov function.* (For the definitions of positive definiteness, negative definiteness and radial unboundedness of a function, refer to, e.g., [27, chap. 5].)

In the stability analysis of the equilibrium $x_e = 0$ of system (1), we will find it useful to employ Lyapunov functions v whose value for a given state vector $w \notin D^n$ is greater than the value for the corresponding saturated state vector sat (w). Specifically, we will make the following assumption.

Assumption (A-1): Assume that for the system (4), there exists a continuous function $v: R^n \to R$ with the following properties:

(i) v is positive definite, radially unbounded, and

$$\begin{aligned} Dv_{(4)}(w(k)) &\triangleq v(w(k+1)) - v(w(k)) \\ &= v(Aw(k)) - v(w(k)) \end{aligned}$$

is negative definite for all $w(k) \in R^n$ (and thus the eigenvalues of A are within the unit circle);

(ii) For all $w \in R^n$ such that $w \notin D^n$, it is true that

$$v(\text{sat}(w)) < v(w) \tag{7}$$

where $D^n \triangleq \{w \in R^n: -1 \le w_i \le 1,\ i = 1,\cdots,n\}$ and sat $(\cdot)$ is defined in (1). ■

An example of a function $v_1: R^2 \to R$ which satisfies (7) is given by $v_1(w) = d_1 w_1^2 + d_2 w_2^2$, $d_1, d_2 > 0$. On the other hand, the function $v_2: R^2 \to R$ given by $v_2(w) = w_1^2 +$

$(2w_1 + w_2)^2$ does not satisfy (7). To see this, consider the point $w = (-0.99, 1.05)^T \notin D^2$ and note that $v_2(\text{sat}(w)) = 1.9405$ and $v_2(w) = 1.845$.

We are now in a position to prove the following result.

Theorem 2.1: If Assumption (A-1) holds, then the equilibrium $x_e = 0$ of the system (1) is globally asymptotically stable.

Proof: Since Assumption (A-1) is true, there exists a positive definite, radially unbounded function v for the system (4) such that (7) is true, which in turn implies that

$$v(\text{sat}(Aw)) \leq v(Aw), \qquad \text{for all } w \in R^n.$$

Also, by (A-1),

$$v(Aw(k)) - v(w(k)) < 0, \qquad \text{for all } w(k) \neq 0.$$

Therefore, along the solutions of the system (1), we have

$$\begin{aligned} Dv_{(1)}(x(k)) &= v(x(k+1)) - v(x(k)) \\ &= v(\text{sat}[Ax(k)]) - v(x(k)) \\ &\leq v(Ax(k)) - v(x(k)) < 0 \end{aligned}$$

for all $x(k) \neq 0$ and $Dv_{(1)}(x(k)) = 0$ if and only if $x(k) = 0$. Therefore, $v(x)$ is positive definite and radially unbounded, and $Dv_{(1)}(x)$ is negative definite for all x. Hence, the equilibrium $x_e = 0$ of the system (1) is globally asymptotically stable. ∎

Remark 1: In particular, for fixed p, $1 \leq p \leq \infty$, let us choose

$$v(w) = \|w\|_p = \left(\sum_{i=1}^{n} |w_i|^p \right)^{1/p}$$

for system (4) and assume that $\|A\|_p < 1$, where $\|A\|_p$ denotes the norm induced by $\|w\|_p$. Under these conditions, (A-1) is true. To see this, note that v is positive definite and radially unbounded, that $v(Aw) = \|Aw\|_p \leq \|A\|_p \|w\|_p < \|w\|_p = v(w)$, and that $\|\text{sat}(w)\|_p < \|w\|_p$, for all $w \in R^n$ such that $w \notin D^n$.

Therefore, the equilibrium $x_e = 0$ of the system (1) is globally asymptotically stable if $\|A\|_p < 1$ for some p, $1 \leq p \leq \infty$.

In the case of digital filters, the above argument holds for *any* type of overflow nonlinearity $\varphi: R \to [-1, 1]$. To see this, let $f(w) = [\varphi(w_1), \cdots, \varphi(w_n)]^T$ and note that in this case $\|f(w)\|_p < \|w\|_p$ for all $w \in R^n$ such that $w \notin D^n$. ∎

In order to generate quadratic form Lyapunov functions which satisfy Assumption (A-1) for systems described by (1), we will find it convenient to utilize the next assumption. (Throughout, when using the term *positive-definite matrix*, we will have in mind a *symmetric* matrix with positive eigenvalues.)

Assumption (A-2): Let $x_s = \text{sat}(x) = [\text{sat}(x_1), \cdots, \text{sat}(x_n)]^T$ for $x \in R^n$ and let $H \in R^{n \times n}$ denote a positive definite matrix. Assume that

$$x_s^T H x_s < x^T H x \tag{8}$$

whenever $x \notin D^n$, $x \in R^n$. ∎

An example of a matrix which satisfies (A-2) is any diagonal matrix with positive diagonal elements. On the other hand, the positive definite matrix H given by

$$H = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

does not satisfy Assumption (A-2). (To see this, refer to the example following Assumption (A-1) by noting that $v_2(x) = x^T H x$.)

The next result gives a *necessary and sufficient* condition for matrices to satisfy Assumption (A-2). This result is very useful in applications.

Lemma 1: An $n \times n$ positive definite matrix $H = (h_{ij})$ satisfies Assumption (A-2) *if and only if*

$$h_{ii} \geq \sum_{j=1, j \neq i}^{n} |h_{ij}|, \qquad i = 1, \cdots, n. \tag{9}$$

Proof: See Appendix. ∎

The following result is a direct consequence of Theorem 2.1.

Corollary 2.1: The equilibrium $x_e = 0$ of the system (1) is globally asymptotically stable, if there exists a matrix H which satisfies (A-2), such that $Q \triangleq H - A^T H A$ is positive definite.

By choosing $v(x) = x^T(k) H x(k)$, the proof follows from Theorem 2.1 ∎

Remark 2: For *linear* system (4), the equilibrium $w = 0$ is globally asymptotically stable if and only if all eigenvalues of A are within the unit circle. Equivalently, the equilibrium $w = 0$ of system (4) is globally asymptotically stable if and only if for every positive definite matrix Q, there exists a positive definite matrix P, such that (cf. [28, theorems 8–17])

$$Q = P - A^T P A. \tag{10}$$

Corollary 2.1 tells us that the equilibrium $x_e = 0$ of the *nonlinear* system (1) is globally asymptotically stable if in addition to the conditions given above (for linear system (4)), Assumption (A-2) is satisfied, i.e., there exists a matrix H which satisfies (A-2) such that $H - A^T H A$ is positive definite. ∎

In the next results, Theorem 2.2, we show that Corollary 2.1 is actually true when Q is only positive semidefinite, still assuming that A is stable.

Theorem 2.2: The equilibrium $x_e = 0$ of the system (1) is globally asymptotically stable, if A is stable and if there exists a matrix H which satisfies (A-2), such that $Q \triangleq H - A^T H A$ is positive semidefinite.

Proof: Let us choose $v(x(k)) = x^T(k) H x(k)$ for the system (1). The function v is clearly positive definite and radially unbounded. Also, since

$$\begin{aligned} Dv_{(1)}(x(k)) &= v(x(k+1)) - v(x(k)) \\ &= [\text{sat}(Ax(k))]^T H [\text{sat}(Ax(k))] \\ &\quad - x^T(k) H x(k) \\ &\leq x^T(k)(A^T H A - H) x(k) \end{aligned}$$

and since $H - A^T HA$ is positive semidefinite, $Dv_{(1)}(x(k))$ is negative semidefinite for all $x(k)$. Therefore, the equilibrium $x_e = 0$ is stable. To show that it is asymptotically stable, we must show that $x(k) \to 0$ as $k \to \infty$.

Let us consider an n consecutive-step iteration for the system (1), from $n_0 \geq 0$ to $n + n_0$. Without loss of generality, assume that the system (1) saturates at $k = l, l \in [n_0, n + n_0)$. In view of (A-2), it follows that

$$\begin{aligned} v(x(l+1)) &= x^T(l+1)Hx(l+1) \\ &= [\text{sat}(Ax(l))]^T H[\text{sat}(Ax(l))] \\ &< [Ax(l)]^T HAx(l) \leq x^T(l)Hx(l) = v(x(l)). \end{aligned}$$

On the other hand, if no saturation occurs during this period, then, using the fact that if $H - A^T HA$ is positive semidefinite, then $H - (A^T)^n HA^n$ is positive definite when A is stable (cf. [21]), we have

$$\begin{aligned} v(x(n+n_0)) &= x^T(n+n_0)Hx(n+n_0) \\ &= [A^n x(n_0)]^T HA^n x(n_0) \\ &= x^T(n_0)(A^T)^n HA^n x(n_0) < x^T(n_0)Hx(n_0) \\ &= v(x(n_0)). \end{aligned}$$

Therefore, we can conclude that for the sequence $\{k: k = 1, 2 \cdots\}$, there always exists an infinite subsequence $\{k_j: j = 1, 2, \cdots\}$, such that $Dv_{(1)}(x(k_j))$ is negative for $x(k_j) \neq 0$, and that $v(x(k)) \leq v(x(k_j))$ for all $k \geq k_j$. Since v is a positive definite quadratic form, it follows that $v(x(k_j)) \to 0$ as $j \to \infty$, and therefore $v(x(k)) \to 0$ as $k \to \infty$. This in turn implies that $x(k) \to 0$ as $k \to \infty$. Thus the equilibrium $x_e = 0$ of (1) is globally asymptotically stable. ■

III. Applications to Digital Filters

Equation (1) may be employed to represent fixed-point digital filters using saturation overflow arithmetic under zero input. This model does not include quantization effects. The existence and non-existence of limit cycles in digital filters (under zero input) due to overflow nonlinearities have been investigated extensively (see, e.g., [18]–[26], [29]–[40]). The types of characteristics considered in these studies include zeroing, two's complement, triangular, saturation, and other types of nonlinearities. Since stable *second-order* direct from digital filters using saturation arithmetic have been shown to be free of limit cycles (cf. [29], [30]), filters of *any order*, endowed with saturation nonlinearities have received special attention. In addition, as pointed out in [26], [35], [36], conditions obtained for the absence of nonlinear oscillations (under zero input) in digital filters with saturation overflow nonlinearities are generally less conservative than corresponding conditions obtained for digital filters with other types of overflow characteristics.

A. Some Existing Results

An early result of Barnes and Fam [18] states that if

$$\|A\|_2 = \sqrt{\lambda_M(A^T A)} < 1 \tag{11}$$

where $\|\cdot\|_2$ denotes the norm of a matrix induced by the l_2 vector norm, A^T represents the transpose of A, and $\lambda_M(A^T A)$ denotes the maximum eigenvalue of $A^T A$, then the digital filter (1) is free of limit cycles. It turns out that this result is true for many other types of overflow nonlinearities. An extension to this result is that the matrix Q in (12) be positive semidefinite, assuming that A is stable, i.e., assuming that every eigenvalue λ_i of A satisfies $|\lambda_i| < 1$,

$$Q = D - A^T DA \geq 0 \tag{12}$$

where D is a diagonal matrix with positive diagonal elements [19]–[21]. This result can not be applied to the case in which the absolute values of some diagonal elements in matrix A are greater than or equal to 1. Another extension to condition (11) is given by

$$\|A\|_p < 1 \qquad \text{for some } p, \qquad 1 \leq p \leq \infty \tag{13}$$

where $\|\cdot\|_p$ denotes the matrix norm induced by the l_p vector norm. (Condition (13) is stated in [24] without proof.)

Another time domain result, which has no obvious relations with any of the results cited above, states that if

$$\rho(|A|) < 1 \tag{14}$$

where $\rho(\cdot)$ denotes the spectral radius and $|A| = (|a_{ij}|)$, then the digital filter (1) is free of limit cycles (cf. [22]–[24]). This result is especially useful for testing a digital filter with lower or upper triangular coefficient matrix.

It is shown by Singh [25], [26], that the frequency-domain condition

$$D + DA(zI - A)^{-1} + [DA(zI - A)^{-1}]^* \geq 0, \qquad \text{for all } |z| = 1 \tag{15}$$

is equivalent to condition (12), where I denotes the $n \times n$ identity matrix, z is a complex variable, and * represents the conjugate transpose. An improvement to condition (15), assuming saturation arithmetic in the digital filters, given by

$$2D + DA(zI - A)^{-1} + [DA(zI - A)^{-1}]^* \geq 0, \qquad \text{for all } |z| = 1 \tag{16}$$

is also due to Singh [26]. Note that in (15) and (16), D is still assumed to be a diagonal matrix with positive diagonal elements and that A is assumed to be a stable matrix.

We note that conditions (11)–(16) constitute also conditions for the global asymptotic stability of the null solutions of the digital filters under investigation (with no external inputs).

B. Digital Filters Using Generalized Overflow Characteristics

Since no limit cycles can exist in a digital filter if its trivial solution is globally asymptotically stable, we can use the results of Section II, to establish the following results for nth order digital filters with saturation arithmetic.

Corollary 3.1:

i) A digital filter described by (1) is free of limit cycles, if Assumption (A-1) is satisfied.

ii) A digital filter described by (1) is free of limit cycles, if A is stable and if there exists a matrix H which satisfies Assumption (A-2), such that $Q \triangleq H - A^T HA$ is positive semidefinite. ■

Remark 3: As pointed out in Remark 1, condition (13) is a special case of Corollary 3.1(i). We note that since in (12), D is assumed to be a diagonal matrix with positive diagonal elements, Corollary 3.1(ii) constitutes a generalization of condition (12). ■

Remark 4: The results given in Corollary 3.1 are in general less conservative than conditions (12), (13), or (15) and appear to have no direct relationships with conditions (14) and (16). However, Corollary 3.1(ii) is considerably easier to apply than condition (16), since the latter involves matrix inversions. In Section IV, we include a specific example which can be analyzed by Corollary 3.1(ii), but not by any of the previous results given by (11)–(16). ■

Remark 5: In [40], it is shown that second-order digital filters given by (1) with

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{17}$$

are free of limit cycles if A is stable and if

$$|a_{11} - a_{22}| \le 2\min(|a_{12}|, |a_{21}|) + 1 - \det(A). \tag{18}$$

This result can also be derived by Corollary 3.1(ii), since under the above conditions, there always exists a matrix H which satisfies Assumption (A-2) with $H - A^T HA$ positive semidefinite (cf. [40]).

We also note that when for a second-order digital filter with

$$A = \begin{pmatrix} 0 & 1 \\ b & a \end{pmatrix}$$

the parameters (a, b) are located within the well-known stability triangle, then condition (18) is automatically satisfied. Thus second-order direct form digital filters with saturation nonlinearities and with matrix A stable, are free of limit cycles. This result was originally established in [29] and [30], using approaches which differ significantly from the present method. ■

In the sequel, we will consider nth-order digital filters described by equations of the form

$$x(k+1) = f[Ax(k)], \qquad k = 0, 1, 2, \cdots \tag{19}$$

where $x(k) \in R^n$, $A \in R^{n \times n}$,

$$f(x) = [\varphi(x_1), \varphi(x_2), \cdots, \varphi(x_n)]^T \tag{20}$$

and $\varphi: R \to [-1, 1]$ is piecewise continuous. We call (19) a *fixed-point digital filter using overflow arithmetic*. For such filters, we will make the following assumption.

Assumption (A-3): Let f be defined as in (20). Assume

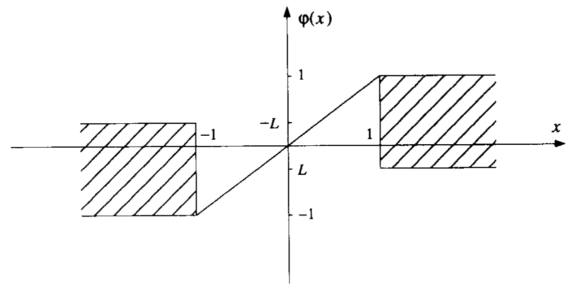

Fig. 1. The generalized overflow nonlinearity described by (23).

that $H \in R^{n \times n}$ is a positive definite matrix and that

$$f(x)^T Hf(x) < x^T Hx \tag{21}$$

for all $x \in R^n$, $x \notin D^n$. ■

In what follows, we will let the function φ in (20) be defined as

$$\varphi(x_i) = \begin{cases} L, & x_i > 1 \\ x_i, & -1 \le x_i \le 1 \\ -L, & x_i < -1 \end{cases} \tag{22}$$

or (see Fig. 1)

$$\begin{cases} L \le \varphi(x_i) \le 1, & x_i > 1 \\ \varphi(x_i) = x_i, & -1 \le x_i \le 1 \\ -1 \le \varphi(x_i) \le -L, & x_i < -1 \end{cases} \tag{23}$$

where $-1 < L \le 1$. We will call (22) or (23) a *generalized overflow characteristic*. Note that when defined in this way, φ includes as special cases the usual types of overflow arithmetic employed in practice, such as zeroing, two's complement, triangular, and saturation overflow characteristics.

To establish our next result, Theorem 3.1, we require the following preliminary result, Lemma 2.

Lemma 2: Assume that f is defined in (20) and φ is given in (22) or in (23). An $n \times n$ positive definite matrix $H = (h_{ij})$ satisfies Assumption (A-3) *if and only if*

$$(1 + L)h_{ii} \ge 2 \sum_{j=1, j \ne i}^{n} |h_{ij}|, \qquad i = 1, \cdots, n. \tag{24}$$

Proof: See Appendix. ■

The overflow arithmetic (23) has also been considered in [31] where it is called *generalized zeroing arithmetic*. We prefer to use the name generalized overflow arithmetic in this paper.

Theorem 3.1: The nth-order digital filter described by (19), in which φ is given in (22) or (23), is free of limit cycles, if A is stable and if there exists a positive definite matrix H which satisfies (24), such that $Q \triangleq H - A^T HA$ is positive semidefinite.

Proof: We can follow the same procedure as in the proof of Theorem 2.2 to prove that under these conditions, the equilibrium $x_e = 0$ of system (19) is globally asymptotically stable. Thus the digital filter described by (19) is free of limit cycles. ■

For the two's complement and triangular overflow characteristics, we have

Lemma 3: An $n \times n$ positive definite matrix $H = (h_{ij})$ satisfies Assumption (A-3) when f represents the two's complement or the triangular arithmetic, *if and only if* H is a diagonal matrix with positive diagonal elements.

Proof: The proof is similar to the proof of Lemma 2. ■

Remark 6: A special case of the overflow characteristics given in (22) is the zeroing characteristic in which $L = 0$. We can also treat the two's complement and the triangular characteristics as special cases of (23) by letting $L \to -1$. I this case, condition (24) will simply mean that matrix H is a *diagonal positive-definite* matrix. ■

Remark 7: For fixed-point digital filters employing two's complement or triangular overflow arithmetic, Theorem 3.1 yields the same result as condition (12), since for these types of arithmetic, the matrix H which satisfies (A-3) must be a *diagonal matrix with positive diagonal elements.* For a digital filter (19) using overflow arithmetic given by (22) or (23), our result in Theorem 3.1 relaxes the matrix D in (12) from a diagonal matrix with positive diagonal elements to a positive definite matrix H which satisfies the condition (24). This should certainly cover a broader class of stable matrices A. ■

For *second-order* digital filters, we have the following Corollary.

Corollary 3.2: Suppose that in a *second-order* digital filter described by (19), $A = (a_{ij})$ is stable and the overflow arithmetic is given by (22) or (23). A sufficient condition for the nonexistence of limit cycles in this digital filter is given by

$$|a_{11} - a_{22}| \leq (1 + L)m + 1 - \det(A) \tag{25}$$

if $1 - \det(A) < M - m$, or by

$$|a_{11} - a_{22}| \leq \frac{1 + L}{2}\sqrt{(1 - \det(A))^2 + 4mM} \tag{26}$$

if $1 - \det(A) \geq M - m$, where $M = \max\{|a_{12}|, |a_{21}|\}$ and $m = \min\{|a_{12}|, |a_{21}|\}$.

Proof: It is proved in [40] that when (18) is satisfied, there exists a 2×2 positive definite matrix H satisfying condition (9), such that $H - A^T HA$ is positive semidefinite, assuming that A is stable. Following the same procedure as in [40], it can be proved that when (25) or (26) is satisfied, there exists a 2×2 positive definite matrix H satisfying condition (24), such that $H - A^T HA$ is positive semidefinite, still assuming that A is stable. ■

Remark 8: Conditions (18), (25), and (26) are applicable only when $a_{12}a_{21} < 0$, for if A in (17) is stable, $a_{12}a_{21} \geq 0$ will guarantee the nonexistence of limit cycles in such digital filters for *any* type of overflow nonlinearities satisfying (20) (cf. [19] for details). A generalization of condition (18) to different types of overflow arithmetic is obtained in [40]. The present result (Corollary 3.2) and the corresponding result given in [40] constitute different conditions which do not cover each other. ■

IV. Examples

To demonstrate the applicability of the present results and compare them with previous results, we now consider several specific examples.

Example 1: For the digital filter (1) considered in [26] with A given by

$$A = \begin{pmatrix} 1 & 2^{-3} \\ -0.1 & 0.9 \end{pmatrix} \tag{27}$$

we have $\|A\|_p > 1$, $p = 1, 2$, or ∞, and $\rho(|A|) > 1$. Furthermore, for this filter it can be verified that there is no diagonal matrix D with positive diagonal elements such that $D - A^T DA$ is positive semidefinite. Therefore, conditions (12)–(15) fail as global asymptotic stability tests for this example.

Hypothesis (A-2) is satisfied for this example by choosing

$$H = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 0.8 \end{pmatrix}. \tag{28}$$

Since

$$Q = H - A^T HA = \begin{pmatrix} 0.092 & 0.00325 \\ 0.00325 & 0.023875 \end{pmatrix}$$

is positive-definite, all conditions of Theorem 2.2 are satisfied and the equilibrium $x_e = 0$ of system (1) with A specified by (27) is globally asymptotically stable. Therefore, this digital filter is free of limit cycles.

Condition (16) can also be used to ascertain that $x_e = 0$ of the present digital filter is globally asymptotically stable. However, application of condition (16) is extremely involved and cumbersome. ■

Example 2: For the digital filter (1) with

$$A = \begin{pmatrix} 0.6 & -0.2 \\ 0.3 & 1.1 \end{pmatrix} \tag{29}$$

it can be verified that conditions (12)–(16) are not satisfied.

Choosing H as in (28), it is easily verified that all conditions of Theorem 2.2 are satisfied. Therefore, the equilibrium $x_e = 0$ of system (1) with A specified by (29) is globally asymptotically stable. Hence, this digital filter is free of limit cycles. ■

It is extremely difficult to apply condition (16) when the order of the system (1) is greater than 2, as in the next example, where $n = 4$. In particular, the inversion of high-order matrices of variables poses formidable obstacles. On the other hand, the application of Theorem 2.2 to high-order systems is not particularly difficult.

Example 3: For the system (1) with A given by

$$A = \begin{pmatrix} -1 & 0 & 0.1 & 0 \\ 0.2 & -0.6 & 0 & 0.8 \\ -0.1 & 0.1 & 0.8 & 0 \\ 0.1 & 0 & 0.1 & -0.5 \end{pmatrix} \tag{30}$$

it can easily be verified that $\|A\|_p > 1$, $p = 1, 2$, or ∞, that $\rho(|A|) > 1$, and that there is no diagonal matrix D with positive diagonal elements such that $D - A^T DA$ is positive semidefinite. Hence, conditions (12)–(15) fail as global asymptotic stability tests for the present example.

Hypothesis (A-2) is satisfied for this example by choosing

$$H = \begin{pmatrix} 1.4 & 0 & -0.2 & 0.4 \\ 0 & 1.6 & 0.2 & -0.4 \\ -0.2 & 0.2 & 3.4 & 0.5 \\ 0.4 & -0.4 & 0.5 & 3 \end{pmatrix}. \quad (31)$$

Since

$$Q = H - A^THA = \begin{pmatrix} 0.026 & 0.161 & -0.003 & 0.077 \\ 0.161 & 1.014 & -0.003 & 0.497 \\ -0.003 & -0.003 & 1.124 & 0.774 \\ 0.077 & 0.497 & 0.774 & 0.906 \end{pmatrix}$$

is positive-definite, all conditions of Theorem 2.2 are satisfied, and the equilibrium $x_e = 0$ of the system (1) with such a coefficient matrix is globally asymptotically stable.

From Theorem 3.1, we see that a fourth-order digital filter described by (19) with A given in (30), when f represents the zeroing arithmetic, is free of limit cycles since H in (31) satisfies

$$h_{ii} > 2 \sum_{j=1, j\neq i}^{n} |h_{ij}|, \qquad i = 1, \cdots, n.$$

Indeed, it is also free of limit cycles when generalized overflow arithmetic specified in (22) or (23) is used with $-0.1333 \leq L \leq 1$. ■

V. Concluding Remarks

We emphasize that (1) describes a large class of discrete-time dynamical systems with saturation nonlinearities (which include important classes of digital filters as special cases). Theorem 2.1 which requires the existence of a function v for system (4), satisfying Assumption (A-1), guarantees the global asymptotic stability of the equilibrium $x_e = 0$ of system (1). The two special forms of the function v considered in this paper are the l_p vector norm and the quadratic form. These are two important forms of Lyapunov functions. There may be other forms of Lyapunov functions for system (4), which satisfy Assumption (A-1) under some other conditions.

The result for global asymptotic stability of the null solution of system (1) established in the present paper (Theorem 2.2) is very general, since it involves necessary and sufficient conditions under which a positive definite matrix can be used to generate quadratic form Lyapunov functions for system (1) (Lemma 1). Our results of Section II can be used directly as criteria for testing the nonexistence of overflow limit cycles in nth-order digital filters using saturation arithmetic.

The generalized overflow characteristics considered in Section III cover the usual types of overflow arithmetic used in practice. Our conditions for testing whether a digital filter, using the generalized overflow characteristics, is free of limit cycles, constitute a generalization of condition (12), originally given in [19]–[21]. We generalize the matrix D in (12) from a diagonal matrix with positive diagonal elements to a positive-definite matrix satisfying condition (24).

Appendix

Since Lemma 1 is a special case of Lemma 2 when $L = 1$, we need to prove only Lemma 2.

Proof of Lemma 2: We first prove this lemma for the overflow arithmetic given in (22). We introduce the following notation. For φ defined in (22), let us denote

$$f(x) = [\varphi(x_1), \cdots, \varphi(x_n)]^T = Ex$$

where $E = \text{diag}(e_1, e_2, \cdots, e_n)$, $e_i = 1$ when $|x_i| \leq 1$, and $e_i = L/|x_i|$ when $|x_i| > 1$.

Then, we have

$$x^THx - f(x)^THf(x) = x^T(H - EHE)x.$$

Sufficiency: Suppose $x = (x_1, x_2, \cdots, x_n)^T$, $|x_k| > 1$ and $|x_i| \leq 1$ for $i \neq k$ ($x \notin D^n$). We have $-1 < e_k < 1$, $e_i = 1$ for $i \neq k$, and therefore,

$$H - EHE = \begin{pmatrix} 0 & \cdots & 0 & h_{1k}(1-e_k) & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & h_{k-1,k}(1-e_k) & 0 & \cdots & 0 \\ h_{k1}(1-e_k) & \cdots & h_{k,k-1}(1-e_k) & h_{kk}(1-e_k^2) & h_{k,k+1}(1-e_k) & \cdots & h_{kn}(1-e_k) \\ 0 & \cdots & 0 & h_{k+1,k}(1-e_k) & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & h_{nk}(1-e_k) & 0 & \cdots & 0 \end{pmatrix}$$

and

$$x^T(H - EHE)x = (1 - e_k)\left(h_{kk}(1 + e_k)x_k^2 + 2 \sum_{i=1, i\neq k}^{n} h_{ik}x_ix_k\right). \quad (32)$$

Note that in the above equation we have used the fact that $h_{ij} = h_{ji}$. From $|x_i| \leq 1$ for $i \neq k$, $|x_k| > 1$, $e_k|x_k| = L$, and $L > -1$, we have

$$(1+L)|x_ix_k| \leq (1+L)|x_k| < (|x_k| + L)|x_k| = (1+e_k)x_k^2.$$

Hence, from (32), we have

$$x^T(H - EHE)x \geq (1 - e_k)\left(h_{kk}(1 + e_k)x_k^2 - 2\sum_{i=1,i\neq k}^{n}|h_{ik}x_i x_k|\right) > (1 - e_k^2)x_k^2\left(h_{kk} - \frac{2}{1+L}\sum_{i=1,i\neq k}^{n}|h_{ik}|\right) \geq 0$$

i.e., $x^T Hx > x^T EHEx = f(x)^T Hf(x)$.

Denote $M = \{1, 2, \cdots, m\}$ for *any* m, $0 < m \leq n$ and $N = \{k_i : 0 < k_i \leq n,\ k_i \neq k_j, \text{ when } i \neq j,\ i \in M\}$. Now suppose that $x = (x_1, x_2, \cdots, x_n)^T$, $|x_k| > 1$ for $k \in N$ and $|x_i| \leq 1$ for $i \notin N$ ($x \notin D^n$). Following the same procedure as above, we have

$$\begin{aligned} x^T(H - EHE)x &= \sum_{k\in N}(1 - e_k)\left(h_{kk}(1 + e_k)x_k^2 + 2\sum_{i=1,i\notin N}^{n} h_{ik}x_i x_k\right) + \sum_{k\in N}\sum_{l\in N,l\neq k} h_{kl}x_k x_l(1 - e_k e_l) \\ &\geq \sum_{k\in N}(1 - e_k)\left(h_{kk}(1 + e_k)x_k^2 - 2\sum_{i=1,i\notin N}^{n}|h_{ik}x_i x_k|\right) + \sum_{k\in N}\sum_{l\in N,l\neq k} h_{kl}x_k x_l(1 - e_k e_l) \\ &> \sum_{k\in N}(1 - e_k^2)x_k^2\left(h_{kk} - \frac{2}{1+L}\sum_{i=1,i\notin N}^{n}|h_{ik}|\right) + \sum_{k\in N}\sum_{l\in N,l\neq k} h_{kl}x_k x_l(1 - e_k e_l) \\ &= \sum_{k\in N}(1 - e_k^2)x_k^2\left(h_{kk} - \frac{2}{1+L}\sum_{i=1,i\neq k}^{n}|h_{ik}|\right) + \frac{2}{1+L}\sum_{k\in N}(1 - e_k^2)x_k^2\sum_{i\in N,i\neq k}|h_{ik}| \\ &\quad + \sum_{k\in N}\sum_{l\in N,l\neq k} h_{kl}x_k x_l(1 - e_k e_l). \end{aligned} \tag{33}$$

The first summation of the right-hand side in (33) is nonnegative, by assumption. Considering the last two terms in (33), by noting that $-1 < e_k < 1$ and $e_k|x_k| = L$ for $k \in N$, and $-1 < L \leq 1$, we have

$$\begin{aligned} &\frac{2}{1+L}\sum_{k\in N}(1 - e_k^2)x_k^2\sum_{i\in N,i\neq k}|h_{ik}| + \sum_{k\in N}\sum_{l\in N,l\neq k} h_{kl}x_k x_l(1 - e_k e_l) \\ &\geq \sum_{k\in N}\sum_{l\in N,l\neq k}(1 - e_k^2)x_k^2|h_{kl}| - \sum_{k\in N}\sum_{l\in N,l\neq k}|h_{kl}x_k x_l|(1 - e_k e_l) \\ &= \sum_{k\in N}\sum_{l\in N,l\neq k}|h_{kl}x_k|(|x_k| - e_k L - |x_l| + e_k L) \\ &= \sum_{k\in N}\sum_{l\in N,l\neq k}|h_{kl}|x_k^2 - \sum_{k\in N}\sum_{l\in N,l\neq k}|h_{kl}x_k x_l| \\ &= \sum_{k\in N}\sum_{l\in N,l>k}|h_{kl}|(x_k^2 + x_l^2) - 2\sum_{k\in N}\sum_{l\in N,l>k}|h_{kl}x_k x_l| \\ &= \sum_{k\in N}\sum_{l\in N,l>k}|h_{kl}|(|x_k| - |x_l|)^2 \geq 0. \end{aligned}$$

Therefore,

$$x^T Hx - f(x)^T Hf(x) = x^T(H - EHE)x > 0$$

for any $x \in R^n$ such that $x \notin D^n$.

This proves the sufficiency.

Necessity: It suffices to show that if (24) does not hold, there always exist some points $x \notin D^n$, such that

$$x^T Hx \leq f(x)^T Hf(x).$$

Suppose that (24) does not hold for $i = k$, i.e.,

$$\delta \triangleq 2\sum_{j=1,j\neq k}^{n}|h_{kj}| - (1 + L)h_{kk} > 0.$$

Let us choose $|x_k| = 1 + \xi$, $\xi > 0$, and $x_i = -\text{sign}(h_{ik}x_k)$, $i \neq k$, where

$$\text{sign}(y) = \begin{cases} 1, & y > 0 \\ 0, & y = 0 \\ -1, & y < 0. \end{cases}$$

Then, $x = (x_1, \cdots, x_n)^T \notin D^n$ and (32) becomes

$$\begin{aligned} x^T(H - EHE)x &= (1 - e_k)\left(h_{kk}(1 + e_k)x_k^2 - 2\sum_{i=1,i\neq k}^{n}|h_{ik}x_k|\right) \\ &= (1 - e_k)|x_k|\left(h_{kk}\xi + (1 + L)h_{kk} - 2\sum_{i=1,i\neq k}^{n}|h_{ki}|\right) \\ &= (1 - e_k)|x_k|(h_{kk}\xi - \delta). \end{aligned}$$

Clearly, when we choose

$$0 < \xi \leq \frac{\delta}{h_{kk}}$$

we have

$$x^T Hx - f(x)^T Hf(x) = x^T(H - EHE)x \leq 0.$$

Note here that $h_{kk} > 0$ since H is positive definite.

This proves the necessity.

For the overflow nonlinearity given in (23), the proof of sufficiency is similar to the proof given above. To prove necessity, we note that for a given L, when $|x_i| > 1$, $\varphi(x_i)$ in (23) may assume *any* value in the crosshatched regions in Fig. 1 including $\pm L$ (which is the case for the arithmetic given by (22)). ■

References

[1] M. Vassilaki, J. C. Hennet, and G. Bitsoris, "Feedback control of linear discrete-time systems under state and control constraints," *Int. J. Contr.*, vol. 47, pp. 1727–1735, June 1988.

[2] A. Benzaouia and C. Burgat, "Regulator problem for linear discrete-time systems with non-symmetrical constrained control," *Int. J. Contr.*, vol. 48, pp. 2441–2451, Dec. 1988.

[3] F. Blanchini, "Feedback control for linear time-invariant systems with state and control bounds in the presence of disturbances," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 1231–1234, Nov. 1990.

[4] Jyh-Horng Chou, "Stabilization of linear discrete-time systems with actuator saturation," *Syst. Contr. Lett.*, vol. 17, pp. 141–144, Aug. 1991.

[5] M. Sznaier and M. J. Damborg, "Control of constrained discrete time linear systems using quantized controls," *Automatica*, vol. 25, 623–628, July 1989.

[6] P. Kapasouris, M. Athans, and G. Stein, "Design of feedback control systems for stable plants with saturating actuators," in *Proc. 27th Conf. on Decision and Control*, Austin, TX, pp. 469–479, Dec. 1988.

[7] Bor-Sen Chen, Ching-Chang Wong, "Robust stabilization of multivariable systems via constrained control: Time domain approach," in *Proc. 26th Conf. on Decision and Control*, Los Angeles, CA, pp. 1292–1297, Dec. 1987.

[8] Bor-Sen Chen, Sin-Syung Wang, "The design of feedback controller with nonlinear saturating actuator: Time domain approach," in *Proc. 25th Conf. on Decision and Control*, Athens, Greece, pp. 2048–2053, Dec. 1986.

[9] P.-O. Gutman and P. Hagander, "A new design of constrained controllers for linear systems," *IEEE Trans. Automat. Contr.*, vol. AC-30, pp. 22–33, Jan. 1985.

[10] N. J. Krikelis and S. K. Barkas, "Design of tracking systems subject to actuator saturation and integrator wind-up," *Int. J. Contr.*, vol. 39, pp. 667–682, Apr. 1984.

[11] A. H. Glattfelder and W. Schaufelberger, "Stability analysis of single loop control systems with saturation and antireset-windup circuits," *IEEE Trans. Automatic Contr.*, vol. AC-28, pp. 1074–1081, Dec. 1983.

[12] R. L. Kosut, "Design of linear system with saturating linear control and bounded states," *IEEE Trans. Automat. Contr.*, vol. AC-28, pp. 121–124, Jan. 1983.

[13] A. N. Michel, Jie Si, and Gune Yen, "Analysis and synthesis of a class of discrete-time neural networks described on hypercubes," *IEEE Trans. Neural Networks*, vol. 2, pp. 32–46, Jan. 1991.

[14] Jian-Hua Li, A. N. Michel, and W. Porod, "Analysis and synthesis of a class of neural networks: Linear systems operating on a closed hypercube," *IEEE Trans. Circuits Syst.*, vol. 36, pp. 1405–1422, Nov. 1989.

[15] J. J. Hopfield and D. W. Tank, "'Neural' computation of decisions in optimization problems," *Biol. Cybern.*, vol. 52, pp. 141–152, 1985.

[16] D. W. Tank and J. J. Hopfield, "Simple 'neural' optimization networks: An A/D converter, signal decision circuit, and a linear programming circuit," *IEEE Trans. Circuits Syst.*, vol. CAS-33, pp. 533–541, May 1986.

[17] R. K. Miller, M. S. Mousa, and A. N. Michel, "Quantization and overflow effects in digital implementations of linear dynamic controllers," *IEEE Trans. Automat. Contr.*, vol. AC-33, pp. 698–704, July 1988.

[18] C. W. Barnes and A. T. Fam, "Minimum norm recursive digital filters that are free of overflow limit cycles," *IEEE Trans. Circuits and Syst.*, vol. CAS-24, pp. 569–574, Oct. 1977.

[19] W. L. Mills, C. T. Mullis, and R. A. Roberts, "Digital filter realizations without overflow oscillations," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-26, pp. 334–338, Aug. 1978.

[20] M. Kawamata and T. Higuchi, "On the absence of limit cycles in a class of state-space digital filters which contains minimum noise realizations," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-32, pp. 928–930, Aug. 1984.

[21] P. P. Vaidyanathan and V. Liu, "An improved sufficient condition for absence of limit cycles in digital filters," *IEEE Trans. Circuits Syst.*, vol. CAS-34, pp. 319–322, Mar. 1987.

[22] P. H. Bauer and E. I. Jury, "A stability analysis of two-dimensional nonlinear digital state-space filters," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 38, pp. 1578–1586, Sept. 1990.

[23] ——, "Stability analysis of multidimensional (m-D) direct realization digital filters under the influence of nonlinearities," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 36, pp. 1770–1780, Nov. 1988.

[24] T. Bose and Mei-Qin Chen, "Overflow oscillations in state-space digital filters," *IEEE Trans. Circuits Syst.*, vol. 38, pp. 807–810, July 1991.

[25] V. Singh, "Realization of two's complement overflow limit cycle free state-space digital filters: A frequency-domain viewpoint," *IEEE Trans. Circuits Syst.*, vol. CAS-33, pp. 1042–1044, Oct. 1986.

[26] V. Singh, "Elimination of overflow oscillations in fixed-point state-space digital filters using saturation arithmetic," *IEEE Trans. Circuits Syst.*, vol. 37, pp. 814–818, June 1990.

[27] R. K. Miller and A. N. Michel, *Ordinary Differential Equations.* New York: Academic, 1982.

[28] K. Ogata, *State Space Analysis of Control Systems.* Englewood Cliffs, NJ: Prentice-Hall, 1967.

[29] I. W. Sandberg, "A theorem concerning limit cycles in digital filters," in *Proc. 7th Ann. Allerton Conf. on Circuit and System Theory*, Univ. of Illinois, Urbana, pp. 63–68, Oct. 1969.

[30] P. M. Ebert, J. E. Mazo, and M. G. Taylor, "Overflow oscillations in digital filters," *Bell Syst. Tech. J.*, vol. 48, pp. 2999–3020, Nov. 1969.

[31] A. N. Willson, Jr., "Limit cycles due to adder overflow in digital filters," *IEEE Trans. Circuit Theory*, vol. CT-19, pp. 342–346, July 1972.

[32] T. A. C. Claasen and W. F. G. Mecklenbräuker, and J. B. H. Peek, "Frequency domain criteria for the absence of zero-input limit cycles in nonlinear discrete-time systems, with applications to digital filters," *IEEE Trans. Circuits Syst.*, vol. CAS-22, pp. 232–239, Mar. 1975.

[33] T. A C. Claasen and L. O. G. Kristiansson, "Necessary and sufficient conditions for the absence of overflow phenomena in a second-order recursive digital filter," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-23, pp. 509–515, Dec. 1975.

[34] D. Mitra, "Large amplitude, self-sustained oscillations in difference equations which describe digital filter sections using saturation arithmetic," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-25, pp. 134–143, Apr. 1977.

[35] I. W. Sandberg, "The zero-input response of digital filters using saturation arithmetic," *IEEE Trans. Circuits Syst.*, CAS-26, pp. 911–915, Nov. 1979.

[36] D. Mita, "Criteria for determining if a high-order digital filter using saturation arithmetic is free of overflow oscillations," *Bell Syst. Tech. J.*, vol. 56, pp. 1679–1699, Nov. 1977.

[37] A. G. Bolton, "A two's complement overflow limit cycle free digital filter structure," *IEEE Trans. Circuit Syst.*, vol. CAS-31, pp. 1045–1046, Dec. 1984.

[38] K. T. Erickson and A. N. Michel, "Stability analysis of fixed-point digital filters using computer generated Lyapunov functions—Part I: Direct form and coupled form filters," *IEEE Trans. Circuits Syst.*, vol. CAS-32, pp. 113–132, Feb. 1985.

[39] V. Singh, "A new realizability condition for limit cycle-free state-space digital filters employing saturation arithmetic," *IEEE Trans. Circuits Syst.*, vol. CAS-32, pp. 1070–1071, Oct. 1985.

[40] J. H. F. Ritzerfeld, "A condition for the overflow stability of second-order digital filters that is satisfied by all scaled state-space structures using saturation," *IEEE Trans. Circuits Syst.*, vol. CAS-36, pp. 1049–1057, Aug. 1989.

[41] G. F. Franklin and J. D. Powell, *Digital Control of Dynamic Systems.* Reading, MA: Addison-Wesley, 1980.

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