Stability Analysis of State-Space Realizations for Two-Dimensional Filters with Overflow Nonlinearities

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***Abstract*— We utilize the second method of Lyapunov to establish sufficient conditions for the global asymptotic stability of the trivial solution of percent nonlinear, shift-invariant 2-D (two-dimensional) systems. We apply this result in the stability analysis of 2-D quarter plane state-space digital filters, which are endowed with a general class of overflow nonlinearities. Utilizing the l_∞ vector norm and the p^{th} power of the l_p vector norm for $1 \leq p < \infty$ as Lyapunov functions, we show that $\| A \|_p < 1$, for some p, $1 \leq p \leq \infty$, constitutes a sufficient condition for the global asymptotic stability of the trivial solution of the 2-D nonlinear digital filters where A denotes the coefficient matrix of the filter operating in its linear range and $\| \cdot \|_p$ denotes the matrix norm induced by the l_p vector norm. Using quadratic form Lyapunov functions, we also establish sufficient conditions for the global asymptotic stability of the null solution of the 2-D digital filters. These results are very general, since they involve necessary and sufficient conditions under which positive definite matrices can be used to generate the quadratic Lyapunov functions for the 2-D digital filters with overflow nonlinearities. We generalize the above results to a class of m-D (multidimensional) digital filters with overflow nonlinearities. To demonstrate the applicability of our results, we consider a specific example.**

I. INTRODUCTION

IN THE IMPLEMENTATION of linear digital filters, signals are usually represented and processed in a finite word-length format. Therefore, such implementations frequently give rise to several kinds of nonlinear effects, such as overflow and quantization. The stability analysis of 2-D (two-dimensional) digital filters subject to such nonlinears has been of increasing interest in recent years [1]–[4].

Since finite word-length realizations of digital filters result in systems that are inherently *nonlinear*, the asymptotic stability of such filters (under zero input) is of great interest in practice. The global asymptotic stability of the null solution guarantees the nonexistence of limit cycles (overflow oscillations) in the realized digital filters. In the present paper, we establish new results for the global asymptotic stability of zero-input 2-D state-space digital filters with overflow nonlinearities. We do not consider quantization effects in the present paper. The stability properties of 1-D digital filters subject to overflow nonlinears have been investigated extensively during the last two decades [7]–[15]. However, a great deal of work that addresses qualitative issues concerning 2-D digital filters endowed with overflow nonlinearities remains to be accomplished.

We consider the quarter plane model of 2-D digital filters described by

$$\begin{bmatrix} x^h(k+1,l) \\ \cdots\cdots\cdots \\ x^v(k,l+1) \end{bmatrix} = f\left(\begin{bmatrix} A_{11} & \vdots & A_{12} \\ \cdots\cdots & \vdots & \cdots\cdots \\ A_{21} & \vdots & A_{22} \end{bmatrix} \begin{bmatrix} x^h(k,l) \\ \cdots\cdots \\ x^v(k,l) \end{bmatrix} \right) \quad k \geq 0,\ l \geq 0 \tag{1}$$

where $x^h \in R^m, x^v \in R^n, A_{11} \in R^{m \times m}, A_{12} \in R^{m \times n}$, $A_{21} \in R^{n \times m}$, $A_{22} \in R^{n \times n}$, and $f(\cdot)$ represents overflow nonlinearities. We also assume for system (1) a finite set of initial conditions, i.e., we assume that two positive integers exist, K and L, such that

$$\begin{aligned} x^h(k,0) &= 0 \text{ for } k \geq K, & x^v(k,0) &= 0 \text{ for } k \geq K, \\ x^h(0,l) &= 0 \text{ for } l \geq L, & x^v(0,l) &= 0 \text{ for } l \geq L. \end{aligned} \tag{2}$$

For the asymptotic stability of the 2-D digital filter (1) with initial conditions (2), a well-known result states that [4], [5] if there exists a diagonal positive definite matrix G such that

$$Q = G - A^T G A \tag{3}$$

is positive definite, then the null solution of (1) is globally asymptotically stable, where

$$A = \begin{bmatrix} A_{11} & \vdots & A_{12} \\ \cdots\cdots & \vdots & \cdots\cdots \\ A_{21} & \vdots & A_{22} \end{bmatrix}. \tag{4}$$

Condition (3) cannot be applied to the case where the absolute values of some of the diagonal elements of matrix A are greater than or equal to one.

In the present paper, we utilize Lyapunov's Second Method to establish new results for the global asymptotic stability of the null solution of the 2-D system (1). One of our results generalizes condition (3) and shows that the matrix G can be relaxed to certain classes of positive definite matrices. We also provide necessary and sufficient conditions under which positive definite matrices can be used to construct quadratic

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form Lyapunov functions with desired properties for the 2-D system (1).

For shift-invariant 2-D digital filters (which are considered in the present paper), the results obtained in [1] require that

$$\rho(|A|) < 1 \tag{5}$$

where $\rho(\cdot)$ denotes the spectral radius and $|A| = (|a_{ij}|)$. Condition (5), which is a special case of (3) (see [18]), is especially useful for testing the 2-D system (1) with lower or upper triangular coefficient matrices.

We call the class of overflow nonlinearities considered herein *generalized overflow characteristics*. These nonlinearities constitute a generalization of the usual types of overflow arithmetic employed in practice. In our approach, we do not characterize these nonlinearities by sector conditions; instead, we characterize them by the range of the nonlinear function representing the overflow arithmetic.

In the next section, we prove several results for the global asymptotic stability of the null solution of 2-D digital filters described by (1) (Theorems 1 and 2, Corollary 1, and Proposition 1). An algorithm for finding the positive definite matrices, which can be used to construct the quadratic form Lyapunov functions for the 2-D digital filters considered herein, is proposed in Section 3. We generalize our results to multidimensional cases in Section 4 (Theorem 3) and Corollary 2. We demonstrate the applicability of the present results by means of a specific example in Section 5. We conclude the present paper in Section 6. Some of the details concerning the proofs of the results of Section 2 are included in the Appendix.

II. Two-Dimensional Digital Filters with Overflow Nonlinearities

Throughout, we will use the notation

$$x(k,l) = \begin{bmatrix} x^h(k,l) \\ \cdots\cdots\cdots \\ x^v(k,l) \end{bmatrix} \tag{6}$$

and

$$x_{11}(k,l) = \begin{bmatrix} x^h(k+1,l) \\ \cdots\cdots\cdots\cdots \\ x^v(k,l+1) \end{bmatrix} \tag{7}$$

for $x^h \in R^m$ and $x^v \in R^n$. Also, we let $D(d)$ denote the set defined by

$$D(d) \triangleq \{(k,l): k+l=d, k \geq 0, l \geq 0\} \tag{8}$$

for some positive integer $d > 0$. (In the context of two-dimensional signal processing, the superscripts h and v suggest the terms "horizontal" and "vertical," respectively, while $D(d)$ suggests indices along a diagonal.)

Consider 2-D shift-invariant systems described by equations of the form

$$\begin{bmatrix} x^h(k+1,l) \\ x^v(k,l+1) \end{bmatrix} = g\left(\begin{bmatrix} x^h(k,l) \\ x^v(k,l) \end{bmatrix}\right), \quad k \geq 0, l \geq 0 \tag{9}$$

or compactly

$$x_{11}(k,l) = g(x(k,l)), \quad k \geq 0, l \geq 0 \tag{10}$$

where g: $R^{m+n} \to R^{m+n}$ is continuous. For such systems, we introduce the following concepts.

Definition 1: A point $x_e \in R^{m+n}$ is called an equilibrium point of the 2-D system (9) *(or equivalently,* (10) *if and only if $x_e = g(x_e)$. Furthermore, if there exists an $r > 0$ such that the open ball $B(x_e, r) \triangleq \{x \in R^{m+n}: \|x - x_e\| < r\}$ contains no equilibrium points of* (9) *other than x_e itself, x_e is called an isolated equilibrium point, where $\|\cdot\|$ denotes any of the equivalent norms on R^{m+n}.* □

We assume, without loss of generality, that $x_e = 0$ and that it is isolated. In the following definitions (Definitions 2 and 3), we assume a finite set of initial conditions as in (2) for system (9).

Definition 2: The equilibrium $x = 0$ of the 2-D system (9) *is said to be stable (in the sense of Lyapunov) if for every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$, such that $\|x(k,l)\| < \varepsilon$ for all $k \geq 0, l \geq 0$, whenever $\|x(k,0)\| < \delta$ for $0 \leq k \leq K$ and $\|x(0,l)\| < \delta$ for $0 \leq l \leq L$ where K and L are specified in* (2). □

Definition 3: The equilibrium $x = 0$ of the 2-D system (9) *is said to be globally asymptotically stable (or asymptotically stable in the large) if:*

(1) It is stable; and

(2) every solution of (9) *tends to the origin as $k + l \to \infty$, i.e.*

$$\lim_{k\to\infty \text{ and/or } l\to\infty} x(k,l) = \lim_{k+l\to\infty} x(k,l) = 0$$

for system (9) *with any initial conditions satisfying* (2). *(Note that in the statement $k + l \to \infty$, we still require that $k \geq 0$ and $l \geq 0$). In this case, the equilibrium $x_e = 0$ is said to be globally attractive..* □

Remark 1: The global asymptotic stability of the equilibrium $x_e = 0$ of (9) implies that system (9) has *one and only one* equilibrium. □

Remark 2: In the present paper, we adapt the methodology of the Lyapunov stability theory of an equilibrium for dynamical systems in the qualitative study of 2-D (and multidimensional) filters. In conventional Lyapunov results, a temporal variable (time) plays a central role. In the present paper, which deals with a class of 2-D (and multidimensional) shift-invariant systems, *time* does not have a role. Instead, the independent variables of interest, are spatial variables. Therefore, in the present context, asymptotic stability provides the following qualitative characterization of 2-D (and multidimensional) systems: (1) Stability of the equilibrium $x_e = 0$ provides a measure of continuity of the state variables $\{x_i(k,l), i = 1,\ldots,m+n; k = 1,2,\ldots; l = 1,2,\ldots\}$ with respect to a finite set of initial states (see (2)); (2) global attractivity of the origin ensures that the magnitudes of the state variables become arbitrarily small when the spatial variables become arbitrary large. □

We will require the following concepts in the sequel.

Definition 4: A continuous function ψ: $[0,r] \to R^+ \triangleq [0,\infty)$ (respectively, ψ: $R^+ \to R^+$), is said to belong to class $\mathcal{K}$, i.e., $\psi \in \mathcal{K}$, if $\psi(0) = 0$ and if ψ is strictly increasing on $[0,r]$ (respectively, on R^+). If ψ: $R^+ \to R^+$, if $\psi \in \mathcal{K}$, and if $\lim_{r\to\infty} \psi(r) = \infty$, then ψ is said to belong to class $\mathcal{KR}$. □

Definition 5: *A continuous function v: $R^{m+n} \to R$ (respectively, v: $B(h) \to R$, where $B(h) \triangleq \{x \in R^{m+n}: \|x\| < h\}$) is said to be positive definite if*

1) $v(0) = 0$, *and*
2) *there exists a $\psi \in \mathcal{K}$ such that $v(x) \geq \psi(\|x\|)$ for all $x \in B(r) = \{x \in R^{m+n} : \|x\| < r\}$, for some $r > 0$.* □

Definition 6: *A continuous function v is said to be negative definite if $-v$ is positive definite.* □

Definition 7: *A continuous function v:$R^{m+n} \to R$ is said to be radially unbounded if*

1) $v(0) = 0$, *and*
2) *there exists a $\psi \in \mathcal{KR}$, such that $v(x) \geq \psi(\|x\|)$ for all $x \in R^{m+n}$.* □

We will employ nonlinearities f:$R^N \to R^N$ to represent overflow effects in 2-D digital filters (1), where N denotes the dimension of the underlying vector space,

$$f(x) = [\varphi(x_1), \ldots, \varphi(x_N)]^T \tag{11}$$

and φ: $R \to [-1, 1]$ is piecewise continuous. We will make use of the notation specified in (4) and we will let

$$f(x) = \begin{bmatrix} f(x^h) \\ \cdots\cdots \\ f(x^v) \end{bmatrix} \text{ for } x = \begin{bmatrix} x^h \\ \cdots \\ x^v \end{bmatrix}$$

Associated with the nonlinear digital filter (1), we will consider linear digital filters given by

$$\begin{bmatrix} w^h(k+1,l) \\ \cdots\cdots\cdots\cdots \\ w^v(k,l+1) \end{bmatrix} = \begin{bmatrix} A_{11} & \vdots & A_{12} \\ \cdots\cdots & \ddots & \cdots\cdots \\ A_{21} & \vdots & A_{22} \end{bmatrix} \begin{bmatrix} w^h(k,l) \\ \cdots\cdots\cdots\cdots \\ w^v(k,l) \end{bmatrix},$$

$$k \geq 0, l \geq 0 \tag{12}$$

where A_{11}, A_{12}, A_{21}, and A_{22} are defined in (1). We will assume a finite set of initial conditions for (12) as in (2), and follow the convention established in (6) and (7) for the vector w.

In analyzing the stability of the equilibrium $x_e = 0$ of the 2-D system (1), we will make use of a class of Lyapunov functions V for the linear system (12).

Assumption (A-1): Assume that for system (12) there exists a continuous function V:$R^{m+n} \to R$ with the following properties:

1) V can be expressed as

$$V(w) = V^h(w^h) + V^v(w^v) \tag{13}$$

where

$$w = \begin{bmatrix} w^h \\ \cdots \\ w^v \end{bmatrix}$$

and V^h: $R^m \to R$ and V^v: $R^n \to R$ are positive definite and radially unbounded. (Thus, V is also positive definite and radially unbounded. See Lemma A.1 in the Appendix.)

Furthermore, along the solutions of (12), V satisfies the condition that

$$\begin{aligned} DV_{(12)}(w(k,l)) &\triangleq V(w_{11}(k,l)) - V(w(k,l)) \\ &= V(Aw(k,l)) - V(w(k,l)) \end{aligned}$$

is negative definite for all $w(k,l) \in R^{m+n}$ (A is defined in (4));

2) For all $w \in R^{m+n}$, it is true that

$$V(f(w)) \leq V(w) \tag{14}$$

where f represents the overflow nonlinearity for (1). □

We are now in a position to establish the following result.

Theorem 1: *If Assumption (A-1) holds, the equilibrium $x = 0$ of the 2-D system (1) is globally asymptotically stable.*

Proof: Since (A-1) is true, there exist positive definite and radially unbounded functions V, V^h, and V^v for system (12), such that (14) is true, which in turn implies that $V(f(Aw)) \leq V(Aw)$ for all $w \in R^{m+n}$. Also, by (A-1), $V(Aw(k,l)) < V(w(k,l))$ for all $w(k,l) \neq 0$. Thus, for the 2-D system (1), we have, using (13) and (14)

$$\begin{aligned} V(x_{11}(k,l)) &= V(f(Ax(k,l)) \leq V(Ax(k,l)) \\ &< V(x(k,l)) \text{ for all } x(k,l) \neq 0 \end{aligned} \tag{15}$$

i.e.

$$\begin{aligned} V(x_{11}(k,l)) &= V^h(x^h(k+1,l)) + V^v(x^v(k,l+1)) \\ &< V(x(k,l)) \end{aligned} \tag{16}$$

for all $x(k,l) \neq 0$.

For any integer $d \geq \max\{K, L\}$, we compute

$$\begin{aligned} &\sum_{(k,l)\in D(d)} V(x(k,l)) > \sum_{(k,l)\in D(d)} V(x_{11}(k,l)) \\ &= \sum_{(k,l)\in D(d)} [V^h(x^h(k+1,l)) + V^v(x^v(k,l+1))] \\ &= \sum_{(k,l)\in D(d)} V^h(x^h(k+1,l)) + V^h(x^h(0,d+1)) \\ &+ \sum_{(k,l)\in D(d)} V^v(x^v(k,l+1)) + V^v(x^v(d+1,0)) \\ &= \sum_{(k,l)\in D(d+1)} V^h(x^h(k,l)) + \sum_{(k,l)\in D(d+1)} V^v(x^v(k,l)) \\ &= \sum_{(k,l)\in D(d+1)} V(x(k,l)). \end{aligned} \tag{17}$$

In the above, we have used the fact that $x^h(0, d+1) = 0$, $x^v(d+1, 0) = 0$, and the positive definiteness of the functions V^h and V^v.

Consider any fixed $\varepsilon > 0$. Since V is radially unbounded, positive definite, there exists a function $\psi_1 \in \mathcal{KR}$ such that $V(0) = 0$ and $V(x) \geq \psi_1(\|x\|)$ for all x satisfying $\|x\| < \varepsilon + 1$. Pick $\delta > 0$ so small that

$$\max_{0\leq d\leq \max\{K,L\}} \left\{ \sum_{(k,l)\in D(d)} V(x(k,l)) \right\} < \psi_1(\varepsilon) \tag{18}$$

whenever $\|x(k,0)\| < \delta$ for $0 \le k \le K$ and $\|x(0,l)\| < \delta$ for $0 \le l \le L$. This is always possible, since K and L are finite (see Lemma A.2 in the Appendix). Then (17) and (18) imply that

$$\sum_{(k,l)\in D(d)} V(x(k,l)) < \psi_1(\varepsilon) \text{ for all } d \ge 0. \tag{19}$$

Hence, $\|x(k,l)\|$ can not reach the value ε for all $k \ge 0$ and $l \ge 0$, since this would imply that

$$V(x(k,l)) \ge \psi_1(\|x(k,l)\|) = \psi_1(\varepsilon) \tag{20}$$

which contradicts (19). Therefore, the equilibrium $x_e = 0$ of the 2-D system (1) is stable (see definition 2).

To complete the proof of the theorem, we must show that for any initial conditions satisfying (2),

$$\lim_{k\to\infty \text{ and/or } l\to\infty} x(k,l) = \lim_{k+l\to\infty} x(k,l) = 0.$$

Since we have overflow nonlinearities in (1), we may assume that $\|x(k,l)\| < C$ for all $k \ge 0$ and $l \ge 0$ for some $C > 0$, without loss of generality. We now define

$$\begin{aligned} DV_{(1)}(x(k,l)) &\triangleq V(x_{11}(k,l)) - V(x(k,l)) \\ &= V(f(Ax(k,l))) - V(x(k,l)). \end{aligned}$$

Equation (15) implies that $DV_{(1)}(x(k,l))$ is negative definite for all $x(k,l) \in R^{m+n}$. Hence, there exists a function $\psi_2 \in \mathcal{K}$ such that $DV_{(1)}(0) = 0$ and $DV_{(1)}(x) \le -\psi_2(\|x\|)$ for all x satisfying $\|x\| < C$. Following the same argument as in (17), we now have, for any $d \ge \max\{K, L\}$,

$$\begin{aligned} &\sum_{(k,l)\in D(d)} DV_{(1)}(x(k,l)) \\ &\quad = \sum_{(k,l)\in D(d)} V(x_{11}(k,l)) - \sum_{(k,l)\in D(d)} V(x(k,l)) \\ &\quad = \sum_{(k,l)\in D(d+1)} V(x(k,l)) - \sum_{(k,l)\in D(d)} V(x(k,l)) \\ &\quad \le - \sum_{(k,l)\in D(d)} \psi_2(\|x(k,l)\|). \end{aligned} \tag{21}$$

Since V is positive definite and radially unbounded and $\psi_2 \in \mathcal{K}$, (21) implies that for (1) with any initial conditions satisfying (2)

$$\lim_{d\to\infty} \left[\sum_{(k,l)\in D(d+1)} V(x(k,l)) - \sum_{(k,l)\in D(d)} V(x(k,l)) \right] = 0.$$

This in turn implies that

$$\lim_{d\to\infty} \sum_{(k,l)\in D(d)} \psi_2(\|x(k,l)\|) = 0.$$

It follows that $\psi_2(\|x(k,l)\|) \to 0$ as $k + l \to \infty$. Therefore, for (1) with any initial conditions satisfying (2), we have that $x(k,l) \to 0$ as $k \to \infty$ and/or $l \to \infty$. □

We will refer to a V function satisfying Theorem 1 as a *Lyapunov function* for the 2-D system (1).

In particular, when we choose the function V as the p^{th} power of the l_p vector norm, $1 \le p < \infty$

$$V(w) = \|w\|_p^p = \sum_{i=1}^{m+n} |w_i|^p \tag{22}$$

we have the following result.

Corollary 1: The equilibrium $x_e = 0$ of the 2-D system (1) *is globally asymptotically stable if*

$$\|A\|_p < 1, \text{ for some } p, 1 \le p < \infty, \tag{23}$$

where $\|\cdot\|_p$ denotes the matrix norm induced by the l_p vector norm.

Proof: It suffices to show that if (23) is true then Assumption (A-1) is satisfied.

Clearly,

$$\begin{aligned} V(w) = \|w\|_p^p = \sum_{i=1}^{m+n} |w_i|^p &= \sum_{i=1}^{m} |w_i^h|^p + \sum_{i=1}^{n} |w_i^v|^p \\ &= \|w^h\|_p^p + \|w^v\|_p^p = V^h(w^h) + V^v(w^v) \end{aligned}$$

where V^h and V^v are defined in the obvious way and are positive definite and radially unbounded. Also, in view of (23), we have

$$\begin{aligned} V(w_{11}(k,l)) &= \|Aw(k,l)\|_p^p \le \|A\|_p^p \|w(k,l)\|_p^p \\ &< \|w(k,l)\|_p^p = V(w(k,l)), \end{aligned}$$

for all $w(k,l) \ne 0$. Thus, Assumption (A-1) part (1) is satisfied. Assumption (A-1) part (2) is also satisfied, since

$$\|f(w)\|_p \le \|w\|_p \tag{24}$$

holds for any p, $1 \le p \le \infty$, and *any* type of overflow nonlineary given in (11).

Remark 1: For 1-D fixed-point digital filters given by

$$x(k+1) = f(Ax(k)), k \ge 0 \tag{25}$$

condition $\|A\|_p < 1$ for some p, $1 \le p \le \infty$, guarantees the global asymptotic stability of the null solution of the digital filter (25) [14], [15]. For 2-D digital filters described by (1), we proved in the above corollary that condition (23) (where $1 \le p < \infty$) guarantees the global asymptotic stability of the equilibrium $x = 0$ for such filters. A special case of (23)

$$\|A\|_2 < 1 \tag{26}$$

has been proved in [4] using a slightly different method. (In [4], (26) was considered as a special case of condition (3).) □

Next, we prove that

$$\|A\|_\infty < 1 \tag{27}$$

is also a sufficient condition for the global asymptotic stability of the null solution of the 2-D system (1), using a different approach from the one used above. Condition (27) does not appear to be readily obtainable from Theorem 1.

Proposition 1: Condition (27) is a sufficient condition for the global asymptotic stability of the equilibrium $x_e = 0$ of the 2-D systems (1).

Proof: Choose a function V for system (1) as

$$V(x) = \|x\|_\infty \triangleq \max_{1\le i\le m+n}\{|x_i|\}.$$

By the definition of the l_∞ vector norm and in view of (24) and (27), we have

$$V(x) = \max\{V^h(x^h), V^v(x^v)\} \tag{28}$$

and

$$\begin{aligned}V(x_{11}(k,l)) &= V(f(Ax(k,l)) = \|f(Ax(k,l))\|_\infty\\ &\le \|Ax(k,l)\|_\infty \le \|A\|_\infty\|x(k,l)\|_\infty\\ &< \|x(k,l)\|_\infty = V(x(k,l))\end{aligned} \tag{29}$$

for all $x(k,l) \ne 0$, where $V^h(x^h) = \|x^h\|_\infty$ and $V^v(x^v) = \|x^v\|_\infty$. Relation (29) can be rewritten as

$$\begin{aligned}&V(x_{11}(k,l)) = \max\{V^h(x^h(k+1,l)), V^v(x^v(k,l+1))\}\\ &< V(x(k,l))\end{aligned} \tag{30}$$

for all $x(k,l) \ne 0, k\ge 0, l \ge 0$.

For $(k,l) \in D(d)$, $d \ge \max\{K,L\}$, let

$$\max_{(k,l)\in D(d)}\{V(x(k,l))\} = V(x(p_d,q_d)) \tag{31}$$

where $p_d + q_d = d$. From (30), we see that

$$\begin{aligned}&V^h(x^h(p_{d+1},q_{d+1}))\\ &\quad\le \max\{V^h(x^h(p_{d+1},q_{d+1})), V^v(x^v(p_{d+1}-1,q_{d+1}+1))\}\\ &\quad< V(x(p_{d+1}-1,q_{d+1}))\end{aligned} \tag{32}$$

and

$$\begin{aligned}&V^v(x^v(p_{d+1},q_{d+1}))\\ &\quad\le \max\{V^h(x^h(p_{d+1}+1,q_{d+1}-1)), V^v(x^v(p_{d+1},q_{d+1}))\}\\ &\quad< V(x(p_{d+1},q_{d+1}-1)).\end{aligned} \tag{33}$$

Clearly, from (28) and (31)–(33) and the fact that $\{(p_{d+1}-1,q_{d+1}),(p_{d+1},q_{d+1}-1)\} \in D(d)$, we now have

$$\begin{aligned}&\max_{(k,l)\in D(d+1)}\{V(x(k,l))\} = V(x(p_{d+1},q_{d+1}))\\ &\quad= \max\{V^h(x^h(p_{d+1},q_{d+1})), V^v(x^v(p_{d+1},q_{d+1}))\}\\ &\quad< \max\{V(x(p_{d+1}-1,q_{d+1})), V(x(p_{d+1},q_{d+1}-1))\}\\ &\quad\le \max_{(k,l)\in D(d)}\{V(x(k,l))\} = V(x(p_d,q_d)),\end{aligned} \tag{34}$$

when $x(k,l) \ne 0$ for all $(k,l) \in D(d)$, $d \ge \max\{K,L\}$.

From (34), the proof of stability of the equilibrium $x_e = 0$ of (1) follows along similar lines as the proof of stability in Theorem 1.

We now prove that the equilibrium $x_e = 0$ of (1) is globally attractive, i.e., for system (1) with any initial conditions satisfying (2), $x(k,l) \to 0$ as $k+l \to \infty$ with $k \ge 0$ and $l \ge 0$.

Relation (29) implies that

$$\begin{aligned}DV_{(1)}(x(k,l)) &\triangleq V(x_{11}(k,l)) - V(x(k,l))\\ &= V(f(Ax(k,l))) - V(x(k,l))\end{aligned}$$

is negative definite for all $x(k,l) \in R^{m+n}$. Hence, there exists a function $\psi \in \mathcal{K}$ such that $DV_{(1)}(0) = 0$ and $DV_{(1)}(x) \le -\psi(\|x\|)$ for all $\|x\| < C$, i.e.

$$\begin{aligned}&V(x_{11}(k,l)) - V(x(k,l))\\ &= \max\{V^h(x^h(k+1,l)), V^v(x^v(k,l+1))\} - V(x(k,l))\\ &\le -\psi(\|x(k,l)\|).\end{aligned} \tag{35}$$

(C is specified in the proof of Theorem 1).

Denote

$$\begin{aligned}&V^s(x^s(p_{d+1},q_{d+1}))\\ &\quad\triangleq \max\{V^h(x^h(p_{d+1},q_{d+1})), V^v(x^v(p_{d+1},q_{d+1}))\}\\ &\quad= V(x(p_{d+1},q_{d+1}))\end{aligned} \tag{36}$$

where

$$s = \begin{cases} h, & \text{if } V^h(x^h(p_{d+1},q_{d+1})) \ge V^v(x^v(p_{d+1},q_{d+1}))\\ v, & \text{if } V^h(x^h(p_{d+1},q_{d+1})) < V^v(x^v(p_{d+1},q_{d+1}))\end{cases}$$

Relations (32) and (33) can be written as

$$V^s(x^s(p_{d+1},q_{d+1})) < V(x(p_{ds},q_{ds})) \tag{37}$$

where

$$p_{ds} = \begin{cases} p_{d+1}-1, & \text{if } s = h\\ p_{d+1}, & \text{if } s = v\end{cases},$$

and

$$q_{ds} = \begin{cases} q_{d+1}, & \text{if } s = h\\ q_{d+1}-1, & \text{if } s = v\end{cases}.$$

Now using (35) and (36) and with p_{ds} and q_{ds} defined above, we have, when $s = h$

$$\begin{aligned}&V(x(p_{d+1},q_{d+1})) - V(x(p_d,q_d))\\ &\le V^h(x^h(p_{ds}+1,q_{ds})) - V(x(p_{ds},q_{ds}))\\ &\le -\psi(\|x(p_{ds},q_{ds})\|)\end{aligned} \tag{38}$$

and when $s = v$

$$\begin{aligned}&V(x(p_{d+1},q_{d+1})) - V(x(p_d,q_d))\\ &\le V^v(x^v(p_{ds},q_{ds}+1)) - V(x(p_{ds},q_{ds}))\\ &\le -\psi(\|x(p_{ds},q_{ds})\|).\end{aligned} \tag{39}$$

Since V is positive definite and radially unbounded, (38) and (39) imply that

$$\lim_{d\to\infty} V(x(p_d,q_d)) = r \tag{40}$$

for system (1) with any initial conditions satisfying (2), where $r \ge 0$. We next prove that r in (40) is in fact zero.

(38)–(40) $\Rightarrow$

$$\lim_{d\to\infty}\psi(\|x(p_{ds},q_{ds})\|) = 0.$$

$\Rightarrow$

$$\lim_{d\to\infty} x(p_{ds},q_{ds}) = 0.$$

$\Rightarrow$

$$\lim_{d\to\infty} V(x(p_{ds},q_{ds})) = 0.$$

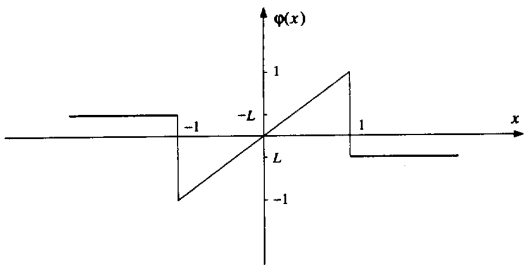

Fig. 1. The generalized overflow nonlinearity described by (41).

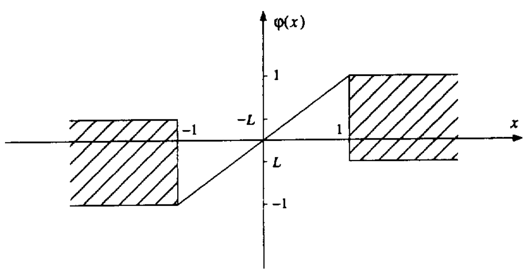

Fig. 2. The generalized overflow nonlinearity described by (42).

Considering (36) and (37), this will, in turn, imply that

$$\lim_{d\to\infty} V(x(p_{d+1}, q_{d+1})) = \lim_{d\to\infty} V^s(x^s(p_{d+1}, q_{d+1})) \leq \lim_{d\to\infty} V(x(p_{ds}, q_{ds})) = 0$$

or equivalently

$$\lim_{d\to\infty} V(x(p_d, q_d)) = 0.$$

Thus, $V(x(k,l)) \to 0$ as $d \to \infty$ for all $(k,l) \in D(d)$, since

$$V(x(p_d, q_d)) = \max_{(k,l)\in D(d)} \{V(x(k,l))\}.$$

Therefore, $x(k,l) \to 0$ as $k + l \to \infty$. It now follows that the equilibrium $x_e = 0$ of the 2-D system (1) is globally asymptotically stable.

In the sequel, we define the function φ in (11) by (see Fig. 1)

$$\varphi(x_i) = \begin{cases} L, & x_i > 1 \\ x_i, & -1 \leq x_i \leq 1 \\ -L, & x_i < -1 \end{cases} \tag{41}$$

or (see Fig. 2)

$$\begin{cases} L \leq \varphi(x_i) \leq 1, & x_i > 1 \\ \varphi(x_i) = x_i, & -1 \leq x_i \leq 1 \\ -1 \leq \varphi(x_i) \leq -L, & x_i < -1 \end{cases} \tag{42}$$

where $-1 \leq L \leq 1$. For given L, when $|x_i| > 1$, $\varphi(x_i)$ in (42) may assume *any* value in the crosshatched region in Fig. 2, including $\pm L$ (which is the case for the arithmetic given by (41)). Note that when defined in this way, φ includes as special cases the usual types of overflow arithmetic employed in practice, such as *zeroing, two's complement, triangular*, and *saturation overflow characteristics*.

The overflow arithmetic (42) has also been considered in [13] where it is called *generalized zeroing arithmetic*. We prefer to use the term *generalized overflow arithmetic* in this paper.

In the following, we will consider a quadratic form Lyapunov function for system (1). In deriving our next result, we make use of the following assumption. (Throughout, when using the term *positive definite matrix*, we will have in mind a *symmetric* matrix with positive eigenvalues.)

Assumption (A-2): Let f be defined as in (11). Assume that there exists a positive definite matrix $H \in R^{N\times N}$ such that

$$f(x)^T H f(x) < x^T H x$$

for all $x \in R^N$, $x \notin D^N \triangleq \{x \in R^N : -1 \leq x_i \leq 1\}$. □

Our next result provides a *necessary and sufficient* condition for matrices to satisfy Assumption (A-2) when f represents the generalized overflow arithmetic. This result is very useful in applications.

Lemma 1: Assume that f is defined in (11) and φ is defined in (41) or (42). An $N \times N$ positive definite matrix $H = (h_{ij})$ satisfies Assumption (A-2) if and only if

$$(1+L)h_{ii} \geq 2 \sum_{j=1, j\neq i}^{N} |h_{ij}|, i = 1, \ldots, N. \tag{43}$$

Proof: See [15]. □

We single out the following special cases of the above lemma:

1. When in (41), $L = 1$, f represents the *saturation overflow nonlinearity* and (43) assumes the form
$$h_{ii} \geq \sum_{j=1, j\neq i}^{N} |h_{ij}|, i = 1, \ldots, N.$$
This case represents a *diagonal dominance condition.*
2. When in (41), $L = 0$, f represents the *zeroing arithmetic* and (43) assumes the form
$$h_{ii} \geq 2 \sum_{j=1, j\neq i}^{N} |h_{ij}|, i = 1, \ldots, N.$$
3. When in (42), $L = -1$, f represents overflow nonlinears, including the *two's complement arithmetic* and *triangular arithmetic*. For such cases, (43) assumes the form
$$h_{ij} = 0, j \neq i, j = 1, \ldots, N,$$
i.e., H is a diagonal matrix with positive diagonal elements.

We are now in a position to prove the following result.

Theorem 2: The equilibrium $x = 0$ of the 2-D digital filter (1) is globally asymptotically stable, if there exist positive definite matrices $H^h \in R^{m\times m}$ and $H^v \in R^{n\times n}$ satisfying Assumption (A-2) (with $N = m$ and $N = n$, respectively), such that

$$Q = H - A^T H A \tag{44}$$

is positive definite, where

$$H = H^h \oplus H^v \triangleq \begin{bmatrix} H^h & \vdots & 0 \\ \cdots\cdots & \cdots & \cdots\cdots \\ 0 & \vdots & H^v \end{bmatrix}.$$

Proof: For (1) we choose the positive definite and radially unbounded Lyapunov function $V(x) = x^T Hx$. Since H^h and H^v satisfy Assumption (A-2), we have

$$\begin{aligned} V(f(x)) &= f(x)^T Hf(x) \\ &= f(x^h)^T H^h f(x^h) + f(x^v)^T H^v f(x^v) \\ &\leq (x^h)^T H^h x^h + (x^v)^T H^v x^v = V(x) \end{aligned}$$

for all $x \in R^{m+n}$. Thus

$$\begin{aligned} V(x_{11}(k,l)) &= f(Ax(k,l))^T Hf(Ax(k,l)) \\ &\leq x(k,l)^T A^T HAx(k,l) \end{aligned} \quad (45)$$

for all $x \in R^{m+n}$. We now have

$$\begin{aligned} V(x_{11}(k,l)) &\leq x(k,l)^T A^T HAx(k,l) \\ &< x(k,l)^T Hx(k,l) = V(x(k,l)) \end{aligned} \quad (46)$$

for all $x(k,l) \neq 0$, since $H - A^T HA$ is positive definite.

The rest of the proof follows along similar lines as the proof of Theorem 1. □

Remark 3: Theorem 2 constitutes a generalization of condition (3). Specifically, we relax the matrix G in (3) from a diagonal positive definite matrix to a positive definite matrix, which is generated from two positive definite matrices satisfying condition (43). This should certainly cover a broader class of coefficient matrices A for 2-D digital filters described by (1) using the generalized overflow nonlinearity with $L > -1$ than condition (3). We also note that we did not use any sector conditions [1], [2] to characterize the overflow nonlinearities. From our present development, it appears that usage of the parameter L given in (41) or (42) to characterize overflow nonlinearities may in some cases be more desirable than usage of sector conditions.

Remark 4: In Section 5, we consider a specific example, which suggests that results provided in Theorem 2 are less conservative than the conditions (5), (23), and (27). Indeed, this example can be analyzed by Theorem 2, but not by conditions (5), (23), and (27). □

Remark 5: In a result that corresponds to Theorem 2 for 1-D digital filters described by (25), we only require that in $Q = H - A^T HA$, matrix H satisfy Assumption (A-2) and that matrix Q be positive semi-definite (under the assumption that A is stable) [15]. A further similar relaxation for the matrix Q in condition (44) has not been achieved, thus far. □

Remark 6: The results developed in the present section can be used directly as criteria for nonexistence of limit cycles (under zero input conditions) of 2-D digital filters described by (1). Very recently, Tzafestas et al. reported new conditions for nonexistence of overflow oscillations of 2-D digital filters subject to overflow nonlinearities [3]. Our results in the present section are more general than the results obtained in [3], since we consider the global asymptotic stability of the equilibrium $x_e = 0$ of 2-D digital filters subject to overflow nonlinearities and since the results in [3] require that *either* A_{11} (or A_{22}) be a scalar *or* the overflow nonlinearity f satisfy $f(x)^T E[x - f(x)] \geq 0$ for all $x \in R^{m+n}$, where E is some positive definite diagonal matrix. □

III. An Algorithm for Determining Matrices H^h and H^v

Theorem 2 does not specify how to determine positive definite matrices H^h and H^v, which satisfy Assumption (A-2). The existence of such positive definite matrices is sufficient for the global asymptotic stability of the null solution of system (1). To apply Theorem 2 and to ascertain the global asymptotic stability of the equilibrium $x = 0$ (or the nonexistence of overflow oscillations) for a given 2-D digital filter with generalized overflow characteristics, it is necessary to determine the positive definite matrices H^h and H^v. For low-order systems, we can usually find H^h and H^v (if they exist) by conducting a search. For high-order systems, such an approach is usually impractical. We suggest in the following an algorithm for determining matrices H^h and H^v for a given 2-D coefficient matrix A and the overflow characteristic (which is characterized by the parameter L, $-1 \leq L \leq 1$).

An Algorithm for Determining Matrices $\mathbf{H^h}$ *and* $\mathbf{H^v}$*:* Suppose A and L are given. Consider an objective function given by

$$J = J(H^h, H^v) = \min_i \lambda_i(Q) = \min_i \lambda_i(H - A^T HA) \quad (47)$$

where $\lambda_i(Q)$ represents the eigenvalues of the matrix Q

$$H = H^h \oplus H^v = \begin{bmatrix} H^h & 0 \\ 0 & H^v \end{bmatrix}$$

and $H^h = [h_{ij}^h] \in R^{m\times m}$ and $H^v = [h_{ij}^v] \in R^{n\times n}$ satisfy the constraints

$$(1+L)h_{ii}^h \geq 2 \sum_{j=1, j\neq i}^{m} |h_{ij}^h|, i = 1, \ldots, m$$

and

$$(1+L)h_{ii}^v \geq 2 \sum_{j=1, j\neq i}^{n} |h_{ij}^v|, i = 1, \ldots, n$$

respectively.

If the maximization of the above objective function results in $J > 0$ for a specified 2-D coefficient matrix A and the parameter L, all conditions of Theorem 2 will be satisfied. Thus, the null solution of the digital filter (1) with such a coefficient matrix and with the generalized overflow nonlineary given by (41) or (42) is globally asymptotically stable. □

The algorithm proposed above is a *nonlinear* programming problem. To determine a solution for this problem may be quite involved. It turns out that we can modify the above algorithm so that its solution will reduce to a standard *linear* programming problem. The disadvantage introduced by this modification is that a solution to the algorithm is not always guaranteed by the maximization of the *modified* objective function J, given below.

It is well known that a measure of a matrix Q, defined by

$$\mu(Q) = \lim_{\theta \to 0^+} \frac{\|I + \theta Q\| - 1}{\theta}$$

where $\|\cdot\|$ denotes a matrix norm and I is the identity matrix, serves as an upper bound for the real parts of the eigenvalues of the matrix Q [16]. When we consider a symmetric matrix Q, $\mu(Q)$ becomes an upper bound for the eigenvalues of Q. In particular, the relationships between $\mu(Q)$ and $\lambda_i(Q)$ is that $\mathrm{Re}\lambda_i(Q) \le \mu(Q)$, and for a symmetric matrix Q, it is $\lambda_i(Q) \le \mu(Q)$. We can transform the nonlinear programming problem stated above into a linear programming problem by choosing the objective function to be a measure of the matrix Q, since some of these measures have *linear* relationships with the entries of the matrix. For example, the measures of matrix $Q = [q_{ij}] \in R^{N \times N}$ induced by the matrix norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are given by

$$\mu_1(Q) = \max_{1 \le j \le N} \{q_{jj} + \sum_{i=1, i \ne j}^{N} |q_{ij}|\}$$

and

$$\mu_\infty(Q) = \max_{1 \le i \le N} \{q_{ii} + \sum_{j=1, j \ne i}^{N} |q_{ij}|\}$$

respectively.

In the present case, since the matrix Q is symmetric, we have $\mu_1(Q) = \mu_\infty(Q)$. Choosing the objective function as

$$J = \mu_1(Q) = \mu_\infty(Q) = \max_{1 \le i \le N} \{q_{ii} + \sum_{j=1, j \ne i}^{N} |q_{ij}|\} \quad (48)$$

we arrive at a linear programming problem. The maximization of J will sometimes result in a set of large eigenvalues for matrix Q. As mentioned earlier, by choosing an objective function J as in (48) and by using the linear programming method to maximize J, we may sometimes not generate a positive definite matrix Q, even if we end up with $J > 0$.

Other alternatives for objective functions, inspired by (48), are

$$J = \min_{1 \le i \le N} \{q_{ii} - \sum_{j=1, j \ne i}^{N} |q_{ij}|\} \quad (49)$$

or

$$J = \min_{1 \le i \le N} \{\sigma_i q_{ii} - \sum_{j=1, j \ne i}^{N} \sigma_j |q_{ij}|\} \quad (50)$$

where $\sigma_i > 0$ for $i = 1, \ldots, N$. The maximization of J in (49) or in (50) will always guarantee a set of large eigenvalues, since in our case Q is symmetric. In particular, if the maximization of the objective function J in (49) or (50) results in $J > 0$, all conditions of Theorem 2 are satisfied. (Under these conditions, $Q = Q^T$ becomes a diagonal dominance matrix with positive diagonal elements. Thus, Q is positive definite. See Lemma A.3 in the Appendix.) However, the objective function $J > 0$ in (49) and (50) may yield conservative results, since these are only sufficient conditions for the matrix Q to be positive definite.

The determination of other objective functions which involve more efficient linear programming problems and whose maximization guarantee the existence of solutions to the problem on hand is under further investigation.

IV. Multidimensional Digital Filters with Overflow Nonlinearities

In this section, we consider m-D (multidimensional or m-dimensional) digital filters described by equations of the form

$$x_I(k_1, \ldots, k_m) = f(Ax(k_1, \ldots, k_m)) \quad (51)$$

where $k_i \ge 0$ for $i = 1, \ldots, m$,

$$x(k_1, \ldots, k_m) = \begin{bmatrix} x^1(k_1, \ldots, k_m) \\ \vdots \\ x^m(k_1, \ldots, k_m) \end{bmatrix}, x^i \in R^{n_i}$$

for $i = 1, \ldots, m$,

$$x_I(k_1, k_2, \ldots, k_m) = \begin{bmatrix} x^1(k_1 + 1, k_2, \ldots, k_m) \\ x^2(k_1, k_2 + 1, \ldots, k_m) \\ \vdots \\ x^m(k_1, k_2, \ldots, k_m + 1) \end{bmatrix}$$

A has compatible dimension and structure, and f represents the overflow nonlinearities defined in (11). We assume a finite set of initial conditions, i.e., we assume that for $i = 1, \ldots, m, x^i(k_1, 0, \ldots, 0) = 0$ for $k_1 \ge K_1$, $x^i(0, k_2, \ldots, 0) = 0$ for $k_2 \ge K_2$, $\ldots, x^i(0, 0, \ldots, k_m) = 0$ for $k_m \ge K_m$, where K_i , $i = 1, \ldots, m$, are finite positive integers.

The definitions for the stability of 2-D system given in Section 2, can be generalized to m-D systems in the obvious way. Furthermore, Theorems 1 and 2 and Corollary 1 and Proposition 1 (applicable to 2-D systems) can also be generalized to m-D systems in the obvious way. The following results constitute generalizations of Theorem 2, Corollary 1, and Proposition 1 to m-D systems described by (51). Their proofs follow along similar lines as in the proofs of the corresponding results for the 2-D case.

Theorem 3: The equilibrium $x = 0$ of the m-D digital filter (51) is globally asymptotically stable if there exist positive definite matrices $H^i \in R^{n_i \times n_i}$ for $i = 1, \ldots, m$, satisfying Assumption (A-2) (with $N = n_i$, $i = 1, \ldots, m$, respectively), such that

$$Q = H - A^T H A \quad (52)$$

is positive definite, where

$$H = H^1 \oplus H^2 \oplus \ldots \oplus H^m \triangleq \begin{bmatrix} H^1 & 0 & \ldots & 0 \\ 0 & H^2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & H^m \end{bmatrix}.$$

Corollary 2: The equilibrium $x_e = 0$ of the m-D system (51) is globally asymptotically stable if

$$\|A\|_p < 1, \text{ for some } p, 1 \le p \le \infty. \quad (53)$$

Remark 7: Theorem 1 can also be generalized to m-D systems in a straightforward manner. In generalizing Theorem

1, we define (analogous to $D(d)$ given in (8)) the hyperplane denoted by $D(M)$

$$D(M) \triangleq \{(k_1, \ldots, k_m) : k_1 + \ldots + k_m = M, k_i \geq 0, \quad i = 1, \ldots, m\} \tag{54}$$

for some integer $M > 0$. The generalization of Theorem 1 (Assumption (A-1)) to m-D systems involves the existence of a Lyapunov function $V : R^P \to R$, where $P = n_1 + \cdots + n_m$, with the following properties:

1) V can be expressed as a sum of functions $V^i : R^{n_i} \to R$, $i = 1, \ldots, m$,
$$V(x) = V^1(x^1) + \ldots + V^m(x^m),$$
2) Each V^i is a function of the partial state x^i only,
3) Every V^i is positive definite and radially unbounded (in the state x^i),
4) The function V satisfies that
$$DV_{(L)}(x(k_1, \ldots, k_m)) \triangleq V(Ax(k_1, \ldots, k_m)) - V(x(k_1, \ldots, k_m))$$
is negative definite for all $x(k_1, \ldots, k_m) \in R^P$, and
5) for all $x \in R^P$, it is true that $V(f(x)) \leq V(x)$, where f represents the overflow nonlinearities for system (51).

It should be noted that for system (51), if (53) is satisfied for some $p, 1 \leq p < \infty$, conditions $(1) - (5)$ above will be satisfied by choosing V as in (22).

V. An Example

To demonstrate the applicability of the present results and compare them with previous results, we now consider a specific example. Specifically, we consider a 3-D (2-2-3) digital filter described by (51) with *saturation* overflow arithmetic and with A given by

$$A = \left[\begin{array}{cc|cc|ccc} 0.8 & -0.2 & 0 & -0.2 & 0 & -0.2 & 0 \\ -0.4 & -0.5 & 0 & -0.2 & 0.2 & 0 & 0 \\ \hline 0.1 & 0.25 & 0.5 & -0.5 & 0.4 & -0.2 & -0.1 \\ 0 & 0 & 0.1 & -0.5 & 0.05 & 0.1 & 0 \\ \hline 0 & -0.3 & 0 & 0.1 & -0.2 & 0.54 & 0.4 \\ 0.15 & 0 & 0 & 0 & -0.1 & 1 & 0.1 \\ 0 & 0.08 & -0.05 & 0 & 0.05 & 0.3 & -0.8 \end{array}\right]. \tag{55}$$

It can be verified that $\|A\|_p > 1$, for $p = 1, 2, \infty$, and $\rho(|A|) > 1$ ($\rho(|A|)$ is defined in (5), and there is no diagonal matrix G with positive diagonal elements such that $G - A^T GA$ is positive definite. Hence, conditions (3), (5), (23) (for $p = 1, 2$), and (27) fail as global asymptotic stability tests for the present example.

According to (43) in Lemma 1 for $L = 1$, we can choose

$$H^1 = \begin{bmatrix} 0.7 & 0.2 \\ 0.2 & 1.2 \end{bmatrix}, H^2 = \begin{bmatrix} 0.4 & -0.15 \\ -0.15 & 0.3 \end{bmatrix}$$

and

$$H^3 = \begin{bmatrix} 1.6 & -0.9 & 0.6 \\ -0.9 & 1.6 & -0.6 \\ 0.6 & -0.6 & 2.1 \end{bmatrix}.$$

We compute

$$H = H^1 \oplus H^2 \oplus H^3$$

and Since Q (shown at bottom of page) is positive definite, all conditions of Theorem 3 are satisfied, and the equilibrium $x_e = 0$ of the 3-D system described by (51) using *saturation* arithmetic with the coefficient matrix given in (55) is globally asymptotically stable.

VI. Concluding Remarks

In the present paper we first established sufficient conditions for the global asymptotic stability of the equilibrium $x = 0$ of 2-D digital filters subject to overflow nonlinears described by (1) (Theorem 1, Corollary 1, Proposition 1, and Theorem 2). The class of overflow nonlinears that we considered herein include as special cases the usual types of overflow arithmetic employed in practice, including zeroing, two's complement, triangular, and saturation overflow characteristics. The stability results developed herein make use of a general class of Lyapunov functions (Theorem 1, i.e., Assumption (A-1)). Two special cases of these Lyapunov functions include quadratic Lyapunov functions (Theorem 2) and l_p vector norms (Corollary 1 and Proposition 1). For quadratic forms, we presented results that enable us to construct the Lyapunov functions (Lemma 1). One of the results presented herein (Theorem 2,) constitutes generalizations to existing stability results (condition (3)) for 2-D digital filters.

Generalizations of the above results to m-D digital filters ($m > 2$) were also presented (Theorem 3, Corollary 2, and Remark 7).

$$Q = H - A^T HA$$

$$= \begin{bmatrix} 0.1480 & 0.0927 & -0.0230 & 0.0580 & 0.0502 & -0.0346 & -0.0380 \\ 0.0927 & 0.6784 & -0.0468 & -0.1015 & 0.0263 & -0.0094 & 0.1510 \\ -0.0230 & -0.0469 & 0.3068 & -0.0770 & -0.0695 & 0.0592 & -0.0565 \\ 0.0580 & -0.1016 & -0.0770 & 0.0920 & 0.1298 & -0.0679 & -0.0195 \\ 0.0503 & 0.0263 & -0.0695 & 0.1298 & 1.4500 & -0.7525 & 0.7323 \\ -0.0346 & -0.0094 & 0.0592 & -0.0679 & -0.7525 & 0.4290 & -0.4773 \\ -0.0380 & 0.1510 & -0.0565 & -0.0195 & 0.7323 & -0.4773 & 0.8400 \end{bmatrix}$$

The results developed herein yield conditions for nonexistence of limit cycles (under zero input conditions) of 2-D and m-D digital filters ($m > 2$) subject to overflow nonlinearities.

To demonstrate the applicability of the present results, we considered a specific example.

APPENDIX

Lemma A.1: Assume that $V^h : R^m \to R$ and $V^v : R^n \to R$ are positive definite functions. Define $V(w) = V^h(w^h) + V^v(w^v)$ for

$$w = \begin{bmatrix} w^h \\ \cdots \\ w^v \end{bmatrix}$$

where $w^h \in R^m$ and $w^v \in R^n$. Then, the function $V : R^{m+n} \to R$ is also a positive definite function.

Proof: Since V^h and V^v are positive definite, there exist functions $\psi_1 \in \mathcal{K}$ and $\psi_2 \in \mathcal{K}$ such that $V^h(w^h) \geq \psi_1(\|w^h\|)$ for all $\|w^h\| \leq r_1$ and $V^v(w^v) \geq \psi_2(\|w^v\|)$ for all $\|w^v\| \leq r_2$, for some positive numbers r_1 and r_2. Let $\|w\| = \max\{\|w^h\|, \|w^v\|\}$, $r = \min\{r_1, r_2\}$, and

$$\psi(\|w\|) = \begin{cases} \min\{\psi_1(\|w^h\|), \psi_2(\|w^h\|)\} & \text{if } \|w^h\| \geq \|w^v\| \\ \min\{\psi_1(\|w^v\|), \psi_2(\|w^v\|)\} & \text{if } \|w^h\| < \|w^v\| \end{cases}.$$

It can easily be shown that ψ is continuous, that $\psi(0) = 0$, and that ψ is strictly increasing on $[0, r]$ Then, $\psi \in \mathcal{K}$, and for any $\|w\| \leq r$

$$V(w) = V^h(w^h) + V^v(w^v) \geq \psi_1(\|w^h\|) + \psi_2(\|w^v\|) \geq \psi(\|w\|).$$

Hence, $V : R^{m+n} \to R$ is also a positive definite function. □

Lemma A.2: Assume that system (1) has a finite set of initial conditions (2). For any $\varepsilon > 0$, we can find a $\delta > 0$ such that

$$\max_{0 \leq d \leq \max\{K,L\}} \left\{ \sum_{(k,l) \in D(d)} V(x(k,l)) \right\} < \varepsilon$$

whenever $\|x(k,0)\| < \delta$ for $0 \leq k \leq K$ and $\|x(0,l)\| < \delta$ for $0 \leq l \leq L$, where the function V is specified in Assumption (A-1).

Proof: For system (12), we define

$$E = \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} \text{ and } F = \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix}$$

and we let $a = \max\{1, \|E\|_m, \|F\|_m\}$, where $\|\cdot\|_m$ denotes the matrix norm induced by the vector norm used herein. Thus, (12) can be written as

$$w(k+1, l+1) = Ew(k, l+1) + Fw(k+1, l), k \geq 0, l \geq 0. \quad \text{(A.1)}$$

Let us consider for (12) (or equivalently, (A.1)) a finite set of initial conditions (2) with $\|x(k,0)\| < \delta_1$ for $0 \leq k \leq K$ and $\|x(0,l)\| < \delta_1$ for $0 \leq l \leq L$. We now claim that for any $d > 0$

$$\max_{(k,l) \in D(d)} \|w(k,l)\| \leq (2a)^{d-1}\delta_1. \quad \text{(A.2)}$$

To prove (A.2), we need only consider $k > 0$ and $l > 0$, since when $k = 0$ or $l = 0$, $\|w(k,l)\| \leq \delta_1$. (A.2) is true for $d = 1$, since we need $k = 0$ or $l = 0$, in $(k,l) \in D(1)$. Suppose (A.2) is true for $d = t$, *i.e.*,

$$\max_{(k,l) \in D(t)} \|w(k,l)\| \leq (2a)^{t-1}\delta_1.$$

Consider now $d = t + 1$, i.e.,$(k,l) \in D(t+1)$ and $\{(k-1, l), (k, l-1)\} \in D(t)$ Since $w(k,l) = Ew(k-1, l) + Fw(k, l-1)$, we have for any $(k,l) \in D(t+1)$, $k > 0$ and $l > 0$.

$$\begin{aligned} \|w(k,l)\| &= \|Ew(k-1,l) + Fw(k,l-1)\| \\ &\leq a(\|w(k-1,l)\| + \|w(k,l-1)\|) \\ &\leq 2a \cdot (2a)^{t-1}\delta_1 = (2a)^t\delta_1. \end{aligned}$$

Therefore, when we confine d to $0 \leq d \leq \max\{K, L\}$, we can find a $\delta_1 = \overline{\delta}_1$ small enough such that each component of $w(k,l)$ will never reach the magnitude of 1 for all $(k,l) \in D(d)$, $0 \leq d \leq \max\{K, L\}$. Thus for system (1) with a finite set of initial conditions (2) and $\|x(k,0)\| < \overline{\delta}_1$ for $0 \leq k \leq K$ and $\|x(0,l)\| < \overline{\delta}_1$ for $0 \leq l \leq L$, we have $\|x(k,l)\| \leq (2a)^{d-1}\overline{\delta}_1$ and each component of $x(k,l)$ will never reach the magnitude of one for all $(k,l) \in D(d)$, $0 \leq d \leq \max\{K, L\}$, since (1) is now operating in the linear range. This in turn implies that for the given initial conditions,

$$\max_{(k,l) \in D(d)} \|x(k,l)\| \leq (2a)^{T-2}\overline{\delta}_1,$$

for all $d \in [0, \max\{K, L\}]$, where $T = \max\{K, L\} + 1$.

For the given $\varepsilon > 0$, we can find a $\delta_2 > 0$ such that $V(x) < \varepsilon/T$ whenever $\|x\| < \delta_2$. Choose

$$\delta = \min\left\{\overline{\delta}_1, \frac{d_2}{(2a)^{T-2}}\right\}.$$

Then, $\|x(k,0)\| < \delta$ for $0 \leq k \leq K$ and $\|x(0,l)\| < \delta$ for $0 \leq l \leq L$ imply that

$$\max_{0 \leq d \leq \max\{K,L\}} \left\{ \max_{(k,l) \in D(d)} \|x(k,l)\| \right\} \leq (2a)^{T-2}\delta \leq \delta_2.$$

This, in turn, implies that

$$\max_{0 \leq d \leq \max\{K,L\}} \left\{ \max_{(k,l) \in D(d)} V(x(k,l)) \right\} < \frac{\varepsilon}{T}.$$

Therefore,

$$\begin{aligned} &\max_{0 \leq d \leq \max\{K,L\}} \left\{ \sum_{(k,l) \in D(d)} V(x(k,l)) \right\} \\ &\leq \max_{0 \leq d \leq \max\{K,L\}} \left\{ (d+1) \max_{(k,l) \in D(d)} V(x(k,l)) \right\} \\ &\leq \max_{0 \leq d \leq \max\{K,L\}} \left\{ T \max_{(k,l) \in D(d)} V(x(k,l)) \right\} \\ &< T \cdot \frac{\varepsilon}{T} = \varepsilon. \end{aligned}$$

This proves the lemma. □

Lemma A.3: *Assume that* $Q = Q^T = [q_{ij}] \in R^{N \times N}$. *If there exist* $\sigma_i > 0$, $i = 1, \cdots, N$, *such that*

$$\min_{1 \le i \le N} \left\{ \sigma_i q_{ii} - \sum_{j=1, j \ne i}^{N} \sigma_j |q_{ij}| \right\} > 0 \tag{A.3}$$

then Q *is positive definite.*

Proof: First, we note that if (A.3) is satisfied, $q_{ii} > 0$ for $i = 1, \ldots, N$. (A.3) then implies that $\overline{Q} = [\overline{q}_{ij}]$ is an M-matrix [17], where

$$\overline{q}_{ij} = \begin{cases} q_{ii}, & i = j \\ -|q_{ij}|, & i \ne j \end{cases}.$$

Since $\overline{Q} = \overline{Q}^T$, from the properties of M-matrices [17], $\overline{Q}$ is also positive definite. For any vector $x \in R^N$, we have

$$\begin{aligned} x^T Q x &= \sum_{i=1}^{N} \sum_{j=1}^{N} x_i q_{ij} x_j = \sum_{i=1}^{N} q_{ii} x_i^2 + \sum_{i=1}^{N} \sum_{j=1, j \ne i}^{N} x_i q_{ij} x_j \\ &\ge \sum_{i=1}^{N} q_{ii} x_i^2 - \sum_{i=1}^{N} \sum_{j=1, j \ne i}^{N} |x_i| |q_{ij}| |x_j| \\ &= |x^T| \overline{Q} |x| > 0, \end{aligned}$$

since $\overline{Q}$ is positive definite, where $|x| = (|x_1|, \ldots, |x_N|)^T$. Therefore, Q is also positive definite. □

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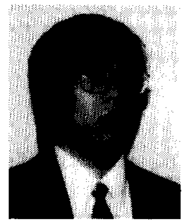

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