# Transactions Briefs

## Stability Analysis of Systems with Partial State Saturation Nonlinearities

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*Abstract*—**Sufficient conditions for the global asymptotic stability of the equilibrium $x_e = 0$ of discrete-time dynamical systems which have saturation nonlinearities on part of the states are established. We utilize a class of positive definite and radially unbounded Lyapunov functions in establishing our results. When using quadratic form Lyapunov functions, our results involve necessary and sufficient conditions under which positive definite matrices can be used to generate Lyapunov functions for the systems considered herein.**

### I. Introduction

In this brief, we will investigate stability properties of systems described by

$$x(k+1) = g[Ax(k)], k = 0, 1, 2, \cdots \tag{1}$$

where $A = [a_{ij}] \in R^{n\times n}$,

$$x(k) = \begin{bmatrix} x_I(k) \\ \cdots\cdots \\ x_{II}(k) \end{bmatrix} \in X_n^{n_1}$$

$$\triangleq \left\{ y = \begin{bmatrix} y_I \\ \cdots \\ y_{II} \end{bmatrix} : y_I \in R^{n_1}, y_{II} \in D^{n_2} \right\}$$

$n = n_1 + n_2$, $D^{n_2} = \{y \in R^{n_2} : -1 \le y_i \le 1, i = 1, \cdots, n_2\}$

$$g(x) = \begin{bmatrix} x_I \\ \cdots\cdots \\ \text{sat}(x_{II}) \end{bmatrix} \text{ for } x = \begin{bmatrix} x_I \\ \cdots \\ x_{II} \end{bmatrix}, x_I \in R^{n_1}$$

$$\text{and } x_{II} \in R^{n_2}$$

$\text{sat}(x_{II}) = [\text{sat}(x_1), \cdots, \text{sat}(x_{n_2})]^T$, and

$$\text{sat}(x_i) = \begin{cases} 1, & x_i > 1 \\ x_i, & -1 \le x_i \le 1 \\ -1, & x_i < -1 \end{cases}.$$

We refer to such systems as *dynamical systems with partial state saturation*. We will say that system (1) is stable if $x_e = 0$ is the only equilibrium of system (1) and $x_e = 0$ is globally asymptotically stable. (Recall that the equilibrium $x_e = 0$ of system (1) is *globally asymptotically stable* if $i$) it is *stable* in the sense of Lyapunov, i.e., for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon)$ such that $\|x(k)\| < \varepsilon$ for all $k = 0, 1, 2, \cdots$, whenever $\|x(0)\| < \delta$ ($\|\cdot\|$ denotes any vector norm), and $ii$) it is *attractive*, i.e., $x(k) \to 0$ as $k \to \infty$.) Also, since we have saturation nonlinearities in (1), it is clear that for any $x(0) \notin X_n^{n_1}$, $x(k) \in X_n^{n_1}$, $k \ge 1$, will always be true. Thus, without loss of generality, we will assume that $x(0) \in X_n^{n_1}$.

Equation (1) represents a class of discrete-time dynamical systems in which symmetric and normalized saturation nonlinearities occur on partial states. Saturation nonlinearities arise very often in the modeling process of dynamical systems. Examples of such systems are common in engineering and include mechanical systems with position and speed limits, electrical systems with limited power supply for the actuators (motors), digital filters implemented in finite wordlength format, and so on. In such cases, there are physical limits for *all* or *part* of the states and the system saturates when it reaches these limits which are usually finite. Qualitative analysis, especially stability analysis, is a fundamental issue in the study of such dynamical systems. Systems with saturation nonlinearities have been investigated by many researchers (see, e.g., [3]–[6], [8]–[12]). In these studies, saturation nonlinearities are assumed for every state in the system which is not always a realistic hypothesis in applications. For example, in the case of the dynamics of a car, variables such as speed and steering angle have finite physical limits, and they saturate when reaching these limits, whereas variables such as yaw velocity and roll velocity are usually assumed to have no constraints. Similar examples exist in many other engineering applications. Therefore, it is natural to consider systems with partial state saturation nonlinearities. To the authors' best knowledge, a stability analysis of systems with saturation nonlinearities on only *part* of the states does not appear to have been addressed thus far. We intend to investigate this problem in the present brief and we will establish a set of sufficient conditions which ensure the global asymptotic stability of the equilibrium $x_e = 0$ of system (1).

In the stability analysis of systems described by (1), some of the first fundamental questions that arise concern the existence and uniqueness of an equilibrium or operating point (which we assume to be the origin, without loss of generality) and the qualitative properties (specifically, stability properties) of such an equilibrium. The condition that the matrix $A$ be stable (i.e., that every eigenvalue $\lambda_i$ of $A$ satisfies $|\lambda_i| < 1$) does not ensure that $x_e = 0$ is a unique equilibrium, and hence, it does not ensure that $x_e = 0$ is asymptotically stable in the large. For example, for system (1) with

$$A = \left[\begin{array}{c:cccc} 0.6 & 0.9 & 0.6 & -0.4 \\ \cdots & \cdots & \cdots & \cdots \\ 0.8 & 0.7 & 0.4 & 0.2 \\ -0.8 & -2.2 & -0.3 & -0.2 \\ -0.3 & -0.2 & -0.2 & 0.7 \end{array}\right],$$

$n_1 = 1$, and $n_2 = 3$, matrix $A$ has eigenvalues $\lambda(A) = -0.3523, 0.7920, 0.6302 \pm 0.7601i$, i.e., $A$ is stable. It is easily verified that in addition to the origin, system (1) with $A$ specified above, has also equilibria at $x_1 = [1.75,\ 1,\ -1,\ -1]^T$ and $x_2 = [-1.75,\ -1,\ 1,\ 1]^T$. Thus, while $x_e = 0$ is certainly asymptotically stable, it is not asymptotically stable in the large.

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## II. Main Results

In establishing our results, we will make use of Lyapunov functions for the linear system corresponding to system (1), given by

$$w(k+1) = Aw(k), \quad k = 0,1,2,\cdots \tag{2}$$

where $A \in R^{n\times n}$ is defined in (1),

$$w(k) = \begin{bmatrix} w_I(k) \\ \cdots\cdots \\ w_{II}(k) \end{bmatrix}, w_I(k) \in R^{n_1}, w_{II}(k) \in R^{n_2}$$

and $n = n_1 + n_2$.

We recall that for a general autonomous system

$$x(k+1) = f(x(k)), \quad k = 0,1,2,\cdots, \tag{3}$$

with $x(k) \in R^n$ and $f\colon R^n \to R^n$,$x_e$ is an equilibrium for (3) if and only if $x_e = f(x_e)$. We assume, without loss of generality that $x_e = 0$ (see, e.g., [7]). Recall also that the equilibrium $x_e = 0$ for system (3) is globally asymptotically stable, if there exists a continuous function $V\colon R^n \to R$ which is positive definite, radially unbounded, and along solutions of (3) satisfies the condition that

$$DV_{(3)}(x(k)) \triangleq V(x(k+1)) - V(x(k))$$

$$= V(f(x(k))) - V(x(k))$$

is negative definite for all $x(k) \in R^n$. The function $V$ is an example of a *Lyapunov function.* (For the definitions of positive definiteness, negative definiteness and radial unboundedness of a function, refer, e.g., see ch. 5 of [7].)

In the stability analysis of the equilibrium $x_e = 0$ of system (1), we will make use of a class of Lyapunov functions $V$ for the linear system (2). Specifically, we will make the following assumption.

**Assumption (A–1)** Assume that for system (2) there exists a continuous function $V\colon R^{n_1+n_2} \to R$ with the following properties: $(i)$ $V$ can be expressed as

$$V(w) = V_I(w_I) + V_{II}(w_{II}) \tag{4}$$

where $V_I\colon R^{n_1} \to R$ and $V_{II}\colon R^{n_2} \to R$ are positive definite and radially unbounded. (Thus, $V$ is also positive definite and radially unbounded.) Furthermore, along the solutions of (2), $V$ satisfies the condition that

$$DV_{(2)}(w(k)) \triangleq V(w(k+1)) - V(w(k))$$

$$= V(Aw(k)) - V(w(k))$$

is negative definite for all $w(k) \in R^{n_1+n_2}$; $(ii)$ It is true that

$$V_{II}(\mathrm{sat}(w_{II})) \le V_{II}(w_{II}) \tag{5}$$

for all $w_{II} \in R^{n_2}$. ■

We are now in a position to establish the following result.

*Theorem 1:* If Assumption (A–1) holds, the equilibrium $x_e = 0$ of system (1) is globally asymptotically stable.

*Proof:* Since (A–1) is true, there exist positive definite and radially unbounded functions $V$, $V_I$, and $V_{II}$ for system (2), such that (4) and (5) are true. We then have for all $w_{II} \in R^{n_2}$, $V_{II}(\mathrm{sat}(w_{II})) \le V_{II}(w_{II})$, and for all $w \in R^{n_1+n_2}$, $V(g(w)) = V_I(w_I) + V_{II}(\mathrm{sat}(w_{II})) \le V_I(w_I) + V_{II}(w_{II}) = V(w)$. Also, by (A–1), $V(Aw(k)) < V(w(k))$ for all $w(k) \ne 0$. Therefore, along the solutions of system (1), we have $DV_{(1)}(x(k)) = V(x(k+1)) - V(x(k)) = V(g[Ax(k)]) - V(x(k)) \le V(Ax(k)) - V(x(k)) < 0$ for all $x(k) \ne 0$ and $DV_{(1)}(x(k)) = 0$ if and only if $x(k) = 0$. Therefore, $V(x)$ is positive definite and radially unbounded, and $DV_{(1)}(x)$ is negative definite for all $x$. Hence, the equilibrium $x_e = 0$ of system (1) is globally asymptotically stable. ■

We will refer to a $V$ function satisfying Theorem 1 as a *Lyapunov function* for system (1). In particular, when we choose the function $V$ as the $p^{th}$ power of the $l_p$ vector norm, $1 \le p < \infty$,

$$V(w) = \|w\|_p^p = \sum_{i=1}^{n_1+n_2} |w_i|^p = \sum_{i=1}^{n_1} |w_{Ii}|^p + \sum_{i=1}^{n_2} |w_{IIi}|^p$$

or as the $l_\infty$ vector norm

$$V(w) = \|w\|_\infty \triangleq \max_{1\le i\le n_1+n_2} \{|w_i|\}$$

we have the following result.

*Corollary 1:* The equilibrium $x_e = 0$ of system (1) is globally asymptotically stable if

$$\|A\|_p < 1, \text{for some } p, \ 1 \le p \le \infty \tag{6}$$

where $\|\cdot\|_p$ denotes the matrix norm induced by the $l_p$ vector norm.

*Proof:* It can be shown that if (6) is true for $1 \le p < \infty$, then Assumption (A–1) is satisfied.

By choosing a function $V$ for system (1) as $V(x) = \|x\|_\infty$, we can prove that $\|A\|_\infty < 1$ is also a sufficient condition for the global asymptotic stability of the null solution of system (1), using a different approach. We omit the details. ■

In the following, we will consider quadratic form Lyapunov functions for system (1). In deriving our next result, we make use of the following assumption. (Throughout, when using the term *positive definite matrix*, we will have in mind a *symmetric* matrix with positive eigenvalues.)

**Assumption (A–2)** Let $y_s = \mathrm{sat}(y)$ for $y \in R^N$ and let $H$ denote a positive definite matrix. Assume that $y_s^T H y_s < y^T H y$ for all $y \in R^N$, $y \notin D^N = \{y \in R^N : -1 \le y_i \le 1, i = 1,\cdots,N\}$. ■

Our next result provides a *necessary and sufficient* condition for matrices to satisfy Assumption (A–2) which is proved in [5]. This result is very useful in applications.

*Lemma 1:* An $N \times N$ positive definite matrix $H = [h_{ij}]$ satisfies Assumption (A–2) if and only if

$$h_{ii} \ge \sum_{j=1, j\ne i}^{N} |h_{ij}|, i = 1,\cdots,N. \quad ■$$

It is clear that system (1) is unstable when $A$ is not a stable matrix. Assuming that $A$ is stable, we can establish the next result.

*Theorem 2:* The equilibrium $x_e = 0$ of system (1) is globally asymptotically stable, if $A$ is stable and if there exist positive definite matrices $H_I \in R^{n_1\times n_1}$ and $H_{II} \in R^{n_2\times n_2}$ with $H_{II}$ satisfying Assumption (A–2) (with $N = n_2$), such that $Q = H - A^T HA$ is positive semidefinite, where

$$H = H_I \oplus H_{II} \triangleq \begin{bmatrix} H_I & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & H_{II} \end{bmatrix}.$$

*Proof:* For (1) we choose the positive definite and radially unbounded Lyapunov function $V(x) = x^T Hx$. Since $H_{II}$ satisfies Assumption (A–2), we have

$$\left(\mathrm{sat}(x_{II})\right)^T H_{II}\ \mathrm{sat}(x_{II}) < x_{II}^T H_{II} x_{II} \tag{7}$$

for all $x_{II} \in R^{n_2}$, $x_{II} \notin D^{n_2}$, which in turn implies that

$$V(g(x)) = g(x)^T H g(x)$$
$$= (x_I)^T H_I x_I + \left(\text{sat}(x_{II})\right)^T H_{II}\ \text{sat}(x_{II})$$
$$\leq (x_I)^T H_I x_I + (x_{II})^T H_{II} x_{II} = V(x)$$

for all $x \in R^{n_1+n_2}$. Thus,

$$DV_{(1)}(x(k)) = V(x(k+1)) - V(x(k))$$
$$= V\left(g[Ax(k)]\right) - V(x(k))$$
$$\leq V(Ax(k)) - V(x(k))$$
$$= x(k)^T A^T H A x(k) - V(x(k)) \leq 0$$

for all $x(k) \neq 0$, since $H - A^T HA$ is positive semidefinite. Therefore, the equilibrium $x_e = 0$ is stable. To show that it is asymptotically stable, we must show that $x(k) \to 0$ as $k \to \infty$.

Let us consider an $n$ consecutive step iteration for the system (1), from $n_0 \geq 0$ to $n+n_0$. Without loss of generality, assume that system (1) saturates at $k = l, l \in [n_0, n+n_0)$. In view of (7), it follows that

$$V(x(l+1)) = x^T(l+1)Hx(l+1) = g(x)^T H g(x)$$
$$= (x_I)^T H_I x_I + \left(\text{sat}(x_{II})\right)^T H_{II}\ \text{sat}(x_{II})$$
$$< (x_I)^T H_I x_I + (x_{II})^T H_{II} x_{II} = V(x),$$

since $x_{II} \notin D^{n_2}$. On the other hand, if no saturation occurs during this period, then, using the fact that if $H - A^T HA$ is positive semidefinite, then $H - (A^T)^n HA^n$ is positive definite when $A$ is stable (*cf.* [13]), we have

$$V(x(n+n_0)) = x^T(n+n_0)Hx(n+n_0)$$
$$= [A^n x(n_0)]^T HA^n x(n_0) = x^T(n_0)(A^T)^n HA^n x(n_0)$$
$$< x^T(n_0)Hx(n_0) = V(x(n_0)).$$

Therefore, we can conclude that for the sequence $\{k : k = 1, 2, \cdots\}$, there always exists an infinite subsequence $\{k_j : j = 1, 2, \cdots\}$, such that $DV_{(1)}(x(k_j))$ is negative for $x(k_j) \neq 0$, and that $V(x(k)) \leq V(x(k_j))$ for all $k \geq k_j$. Since $V$ is a positive definite quadratic form, it follows that $V(x(k_j)) \to 0$ as $j \to \infty$, and therefore $V(x(k)) \to 0$ as $k \to \infty$. This in turn implies that $x(k) \to 0$ as $k \to \infty$. Thus, the equilibrium $x_e = 0$ of (1) is globally asymptotically stable. ■

*Remark 1:* If $n_1 = 0$ and $n_2 = n$, system (1) reduces to

$$x(k+1) = \text{sat}[Ax(k)], k = 0, 1, 2, \cdots. \tag{8}$$

In this case, all of our results established above reduce to the results in [4]. If $n_1 = n$ and $n_2 = 0$, system (1) reduces to (2). In this case, results (Theorem 1 and Corollary 1) established herein reduce to well-known sufficient conditions for the stability of linear systems (2) (*cf.*, e.g., [7]). ■

## III. An Example

To demonstrate the applicability of the present results, we now consider a specific example. Specifically, we consider a fifth-order system described by (1) in which $A$ is stable and is given by

$$A = \begin{bmatrix} 0.4 & -0.2 & \vdots & 0 & -0.2 & 0.5 \\ -0.5 & -0.5 & \vdots & 0.2 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & -0.3 & \vdots & -0.2 & 0.3 & 0.4 \\ 0.1 & 0 & \vdots & -0.1 & 0.8 & 0.1 \\ 0 & 0.1 & \vdots & 0.05 & 0.3 & -0.8 \end{bmatrix} \tag{9}$$

where $n_1 = 2$ and $n_2 = 3$. It can be verified that $\|A\|_p > 1$, for $p = 1, 2, \infty$. Hence, condition (6) (for $p = 1, 2, \infty$) fails as a global asymptotic stability test for the present example.

We try to apply Theorem 2. We can choose

$$H_I = \begin{bmatrix} 0.5 & 0.6 \\ 0.6 & 1.3 \end{bmatrix},$$

and according to Lemma 1, we can choose

$$H_{II} = \begin{bmatrix} 1.6 & -0.9 & 0.6 \\ -0.9 & 1.6 & -0.6 \\ 0.6 & -0.6 & 2.1 \end{bmatrix}.$$

We compute $H = H_I \oplus H_{II}$, and $Q = H - A^T HA$. Since $Q$ is positive definite, all conditions of Theorem 2 are satisfied, and the equilibrium $x_e = 0$ of the system described by (1) with coefficient matrix given in (9) is globally asymptotically stable. ■

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