Global Output Convergence of a Class of Continuous-Time Recurrent Neural Networks With Time-Varying Thresholds

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***Abstract*—This paper discusses the global output convergence of a class of continuous-time recurrent neural networks (RNNs) with globally Lipschitz continuous and monotone nondecreasing activation functions and locally Lipschitz continuous time-varying thresholds. We establish one sufficient condition to guarantee the global output convergence of this class of neural networks. The present result does not require symmetry in the connection weight matrix. The convergence result is useful in the design of recurrent neural networks with time-varying thresholds.**

***Index Terms*—Global output convergence, Lipschitz continuity, Lyapunov diagonal semistability, neural networks, time-varying threshold.**

I. Introduction

IN THIS PAPER, we consider a class of continuous-time recurrent neural networks (RNNs) given by

$$\frac{dx_i(t)}{dt} = \sum_{j=1}^{n} w_{ij} g_j(x_j(t)) + u_i(t)$$
$$x_i(0) = x_{i0}, \quad i = 1, 2, \ldots, n$$

or, equivalently, in matrix format given by

$$\frac{dx}{dt} = Wg(x(t)) + u(t), \quad x(0) = x_0 \tag{1}$$

where $x = (x_1, x_2, \ldots, x_n)^T \in R^n$ is the state vector, $W = [w_{ij}] \in R^{n\times n}$ is a constant connection weight matrix, $u(t) = (u_1(t), u_2(t), \ldots, u_n(t))^T \in R^n$ is a nonconstant input vector function defined on $[0, +\infty)$ which is called the time-varying threshold, $g(x) = (g_1(x_1), g_2(x_2), \ldots, g_n(x_n))^T$ is a nonlinear vector-valued activation function from R^n to R^n, and $y = g(x)$ is called the output of the network (1). When $u(t)$ is a constant vector threshold, the RNN model (1) has been applied to content-addressable memory (CAM) problems in [8] and [9], and is also a subject of study in [1] and [16]. Recently, the RNN model (1) has been widely applied in solving various optimization problems such as linear programming problem [18], [19], shortest path problem [21], sorting problem [20], and assignment problem [17], [22]. This class of neural networks has been demonstrated for easy implementation using electronic circuits.

The RNN model (1) is different from the well-known Hopfield neural networks which have been used in some optimization problems, e.g., [4], [10], and [15]. In some applications of neural networks (e.g., CAM), the convergence of the network in the state space is a basic requirement [13], while in other applications (e.g., some optimization problems), only the convergence in the output space may be required [17], [18], [20]–[22]. Recently, global asymptotic stability and global exponential stability of the Hopfield neural networks have received attention, e.g., [2]–[4], [7], [11], [12], [14], and [23]. Within the class of sigmoidal activation functions, it was proved that negative semidefiniteness of the symmetric connection weight matrix of a neural network model is necessary and sufficient for absolute stability of the Hopfield neural networks [3]. The absolute stability result was extended to absolute exponential stability in [11]. Within the class of globally Lipschitz continuous and monotone nondecreasing activation functions, a series of papers (see, e.g., [4], [12], and [23], and references cited therein) generalized stability conditions and/or conditions on the permitted classes of activation functions as well as the types of stability (absolute, asymptotic, exponential). For the RNN model (1) with constant threshold, which is actually a special case of the general CAM network in [5] (cf. [5, eq. (13)]), a convergence result can easily be obtained that requires symmetry in the connection weight matrix. Convergence of the RNN model (1) with time-varying thresholds has not yet been investigated.

There are several reasons for studying RNNs with time-varying thresholds. First, as mentioned in [7], time-varying thresholds can drive quickly $x(t)$ to some desired region of activation space. Second, in some RNNs for optimization, it is required that their thresholds vary over time to ensure the feasibility and optimality of the solution, as elaborated in [17]–[22]. Third, as the thresholds are also adaptive parameters, the convergence issue arises for online learning of the RNNs. Fourth, the thresholds can be considered as external inputs which are usually time varying. Fifth, in a cascade neural network, its inputs are the outputs of the previous layer, which are usually time varying. For the RNN model considered in this paper, due to the time-varying threshold $u(t)$, (1) is a nonautonomous differential equation. Moreover, it does not contain linear terms as those in the Hopfield networks. Hence, the dynamic structure of this class of neural networks is different from that

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of the Hopfield models. Since different structures of differential equations usually results in quite different dynamic behaviors, the convergence of (1) is expected to be quite different from those of the Hopfield models. Note that outputs of (1) represent the optimal solutions of the optimization problems in [17], [18], and [20]–[22], output convergence of (1) is desirable and necessary.

This paper investigates the global output convergence of the RNN model (1) with globally Lipschitz continuous and monotone nondecreasing activation functions and locally Lipschitz continuous time-varying thresholds. We establish one sufficient condition for the global output convergence of RNN model (1). The present result extends those in [17], [18], and [20]–[22] to more general cases of connection weight matrix. As a consequence, the present result expands the application domain of the RNN model (1).

The remainder of this paper is organized as follows. In Section II, some preliminaries on recurrent neural networks are presented. Convergence results are developed in Section III. An illustrative example is given in Section IV. Finally, we make concluding remarks in Section V.

II. Assumptions and Preliminaries

We assume that the function $g(\cdot)$ in (1) belongs to the class of globally Lipschitz continuous and monotone nondecreasing activation functions; that is, for $g_i(\cdot)$, there exists $\bar{\ell}_i > 0$ such that

$$0 \le \frac{g_i(\theta) - g_i(\rho)}{\theta - \rho} \le \bar{\ell}_i$$

$\forall\, \theta, \rho \in R$, and $\theta \ne \rho$, $i = 1, 2, \ldots, n$. It should be noted that such activation functions may not be bounded. There are many frequently used activation functions that satisfy this condition, for example, $1/(1+e^{-\theta})$, $(2/\pi)\arctan(\theta)$, $\max(0,\theta)$ and $(|\theta + 1| - |\theta - 1|)/2$, where $\theta \in R$. We assume that the time-varying thresholds $u_i(t)$ are locally Lipschitz continuous and satisfy the following conditions:

$$\lim_{t\to+\infty} u_i(t) = \overline{u}_i \tag{2}$$

where $\overline{u}_i$ are some constants, $i = 1, 2, \ldots, n$, i.e., we assume that $\lim_{t\to+\infty} u(t) = \overline{u}$.

Lemma 2.1 (Lemma 1 in [23]): Let $g(\cdot)$ be a globally Lipschitz continuous and monotone nondecreasing activation function, then

$$\int_v^u [g_i(s) - g_i(v)]\, ds \ge \frac{1}{2\bar{\ell}_i}[g_i(u) - g_i(v)]^2 \quad \forall\, v, u \in R \text{ and } i = 1, 2, \ldots, n.$$

For the purpose of our next lemma, we denote

$$\underline{g}_i = \inf_{-\infty<s<+\infty}(g_i(s)) \quad\text{and}\quad \bar{g}_i = \sup_{-\infty<s<+\infty}(g_i(s))$$

where $\underline{g}_i$ and $\bar{g}_i$ may take $-\infty$ and $+\infty$, respectively. Let

$$\begin{aligned} V_1 &= \{v \mid \underline{g}_i \le v_i \le \bar{g}_i \quad i = 1, 2, \ldots, n\} \\ V_2 &= \{v \mid v \text{ satisfies } Wv + \overline{u} = 0\} \\ V &= V_1 \cap V_2. \end{aligned} \tag{3}$$

Lemma 2.2: If V in (3) is not an empty set, there exists at least one constant vector $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T \in R^n$ such that $Wg(x^*) + \overline{u} = 0$

Proof: Since V in (3) is not an empty set, there exists some $v^* = (v_1^*, v_2^*, \ldots, v_n^*)^T$ such that $\underline{g}_i \le v_i^* \le \bar{g}_i$ and $Wv^* + \overline{u} = 0$, $i = 1, \ldots, n$. By the continuity of g_i, there must exist at least one constant x_i^* such that

$$g_i(x_i^*) = v_i^*, \quad i = 1, \ldots, n.$$

As a result, $Wg(x^*) + \overline{u} = 0$ by noting $Wv^* + \overline{u} = 0$. This completes the proof.

Definition 2.1: The RNN model (1) is said to be globally output convergent if, given any $x_0 \in R^n$, there exists a constant vector $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T$ such that

$$\lim_{t\to+\infty} g_i(x_i(t)) = g_i(x_i^*), \quad i = 1, 2, \ldots, n.$$

It should be noted that in Definition 2.1, $g(\eta)$ may be different from $g(\zeta)$ for two different initial condition η_0 and ζ_0. Global convergence in the present paper is in the sense that each nonequilibrium solution of (1) converges to an equilibrium state of (1) (cf. [5] and [13]). Global (state) convergence is usually shown through the use of an energy function whose value decreases monotonically along nonequilibrium solutions of a neural network. Such results guarantee the nonexistence of nonconstant periodic solutions as well as chaotic solutions in the network. However, due to the squashing effects of the activation function $g(\cdot)$, especially when $g(\cdot)$ is given by a hard-limiter type activation function, output convergence of RNN (1) (which is studied in this paper) may not imply state convergence. When RNN (1) is applied to optimization problems, it is the output, not the state, that represents the optimal solutions [17], [18], [20]–[22]. It is, therefore, an important issue to analyze the output convergence of RNNs when state convergence analysis is not available (as in the present case). We note that the state convergence analysis of RNN model (1) may be very difficult to achieve due to the use of time-varying threshold $u(t)$.

Definition 2.2: An $n \times n$ matrix A is said to be Lyapunov diagonally semistable if there exists a diagonal matrix $P = \mathrm{diag}(p_1, p_2, \ldots, p_n)$ with $p_i > 0$ such that $PA + A^TP \le 0$.

In the sequel, $\lambda_{\max}(A)$ is the maximum eigenvalue of A, $\|\cdot\|$ is the norm of a vector or a matrix, $\|x\| = \sqrt{x^Tx}$ for a vector x, and $\|A\| = \sqrt{\lambda_{\max}(A^TA)}$ for a matrix A. Let $P = \mathrm{diag}(p_1, p_2, \ldots, p_n)$ and $p_{\max} = \max\{p_1, p_2, \ldots, p_n\}$.

III. Convergence Analysis

In this section, we will establish the global output convergence of the RNN model (1). We first prove the following lemma that pertains to the existence and uniqueness of solution of the RNN model (1).

Lemma 3.1: If (2) is satisfied and there exists a constant vector $x^* \in R^n$ such that $Wg(x^*) + \overline{u} = 0$, then, given any $x_0 \in R^n$, the RNN model (1) has a unique solution $x(t; x_0)$ defined on $[0, +\infty)$.

Proof: Let $\tilde{u}(t) = u(t) - \overline{u}$ and $z = (z_1, z_2, \ldots, z_n)^T = x - x^*$. Then the RNN model (1) can be transformed into the following equivalent system:

$$\frac{dz}{dt} = WF(z) + \tilde{u}(t), \quad z(0) = z_0 = x_0 - x^* \tag{4}$$

where $F(z) = (f_1(z_1), \ldots, f_n(z_n))^T = g(z + x^*) - g(x^*)$ is a globally Lipschitz continuous and monotone nondecreasing activation function and $F(0) = 0$. Let $\Phi(z,t) = WF(z) + \tilde{u}(t)$. Since $u(t)$ is locally Lipschitz continuous and g is globally Lipschitz continuous, we see that $\tilde{u}(t)$ is locally Lipschitz continuous and F is globally Lipschitz continuous. As a result, $\Phi(z,t)$ is locally Lipschitz continuous. Based on the well-known Existence and Uniqueness Theorem for solutions of ordinary differential equations [6], for any given initial condition z_0, there is a unique solution to (4) in some time interval $[0, t^*(z_0))$ where $t^*(z_0) > 0$ or $t^*(z_0) = +\infty$ such that $[0, t^*(z_0))$ is the maximal right existence interval of the solution $z(t)$.

Let $T_0 \in (0, +\infty)$ be any finite time such that $z(t)$ is a solution of system (4) with any fixed initial point z_0 for $t \in [0, T_0)$. In what follows, we will give an estimate of the solution $z(t)$ for $t \in [0, T_0)$.

Since F is globally Lipschitz continuous and monotone nondecreasing and $F(0) = 0$, there exists a positive number $\ell > 0$ such that $\|F(z)\| \le \ell \|z\|, \forall\, z \in R^n$. Therefore

$$\begin{aligned}\|\Phi(z,t)\| &= \|WF(z) + \tilde{u}(t)\| \\ &\le \|W\|\|F(z)\| + \|\tilde{u}(t)\| \le \ell\|W\|\|z\| + \|\tilde{u}(t)\|.\end{aligned}$$

Meanwhile, noticing that $\tilde{u}(t)$ is continuous on $[0, +\infty)$ and $\lim_{t\to+\infty} \tilde{u}(t) = 0$ we have that there exists a positive number $\xi > 0$ such that $\|\tilde{u}(t)\| \le \xi, \forall\, t \in [0, +\infty)$. Hence, $\|\Phi(z,t)\| \le \ell\|W\|\, \|z\| + \xi$. By integrating both sides of (4) from $t = 0$ to $t < T_0$, we obtain

$$z(t; z_0) = z_0 + \int_0^t \Phi(z(\tau; z_0), \tau) d\tau, \quad \forall\, t \in [0, T_0).$$

Thus, we have for all $t \in [0, T_0)$

$$\begin{aligned}\|z(t; z_0)\| &\le \|z_0\| + \int_0^t (\xi + \ell\|W\|\, \|z(\tau; z_0)\|) d\tau \\ &\le (\|z_0\| + T_0\xi) + \ell\|W\| \int_0^t \|z(\tau; z_0)\| d\tau.\end{aligned}$$

According to Gronwall's lemma [6], it follows that

$$\begin{aligned}\|z(t; z_0)\| &\le (\|z_0\| + T_0\xi) \exp(\ell\|W\|t) \\ &\le (\|z_0\| + T_0\xi) \exp(\ell\|W\|T_0) < +\infty \\ &\forall\, t \in [0, T_0).\end{aligned}$$

Hence, the solution $z(t; z_0)$ will be bounded for $t \in [0, t^*(z_0))$ if $T_0 \triangleq t^*(z_0)$ is finite. Therefore, by virtue of the continuation theorem for the solutions of ordinary differential equations [6], we can conclude that the solution $z(t; z_0)$ exists in the time interval $t \in [0, +\infty)$ and the solution is unique on $[0, +\infty)$. This completes the proof of the lemma.

Remark 3.1: According to Lemma 3.1, given any $x_0 \in R^n$, the RNN model (1) has a unique state solution $x(t; x_0)$ defined on $[0, +\infty)$ if (2) is satisfied and there exists a constant vector $x^* \in R^n$ such that $Wg(x^*) + \overline{u} = 0$. Moreover, the RNN model (1) can be transformed into the equivalent system (4) where $z(t)$ is defined on $[0, +\infty)$

We now establish the main result of the present paper.

Theorem 3.1: Suppose that W is Lyapunov diagonally semistable (i.e., there exists a diagonal matrix $P = \mathrm{diag}(p_1, \ldots, p_n)$ with $p_i > 0$ such that $PW + W^TP \le 0$) and

$$|u_i(t) - \overline{u}_i| \le \alpha_i \exp(-\beta_i t) \tag{5}$$

where $\alpha_i > 0$ and $\beta_i > 0$, $i = 1, 2, \ldots, n$. If there exists a vector $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T \in R^n$ such that $Wg(x^*) + \overline{u} = 0$, then the output vector of the RNN model (1) is globally convergent; that is, given any $x_0 \in R^n$, there exists a vector $\bar{x}^* = (\bar{x}_1^*, \bar{x}_2^*, \ldots, \bar{x}_n^*)^T \in R^n$ such that

$$\lim_{t\to+\infty} g_i(x_i(t)) = g_i(\bar{x}_i^*), \quad i = 1, 2, \ldots, n$$

where $\bar{x}^*$ satisfies $Wg(\bar{x}^*) + \overline{u} = 0$ and $g(\bar{x}^*)$ may be different from $g(x^*)$.

Proof: Based on the conditions in Theorem 3.1 and Remark 3.1, we only need to consider system (4). Let $\alpha_{\max} = \max_{1\le i\le n}(\alpha_i)$ and $\beta_{\min} = \min_{1\le i\le n}(\beta_i)$. Then, $\|\tilde{u}(t)\| \le \sqrt{n}\alpha_{\max}\exp(-\beta_{\min}t)$. We first prove that $\|F(z(t))\|$ is bounded for $[0, +\infty)$. Since W is Lyapunov diagonally semistable, there exists a diagonal matrix $P = \mathrm{diag}(p_1, p_2, \ldots, p_n)$ with $p_i > 0$ such that $PW + W^TP \le 0$. Consider a function

$$E(t) = \sum_{i=1}^n p_i \int_0^{z_i(t)} f_i(s) ds. \tag{6}$$

Computing the time derivative of $E(t)$ along the trajectory $z(t)$ of system (4) yields

$$\begin{aligned}\frac{dE(t)}{dt} &= F(z)^T P(WF(z) + \tilde{u}(t)) \\ &= F(z)^T [PW]^S F(z) + F(z)^T P\tilde{u}(t) \\ &\le \lambda_{\max}([PW]^S)\|F(z)\|^2 \\ &\quad + p_{\max}\|F(z)\|\, \|\tilde{u}(t)\| \quad \forall\, t \in [0, +\infty)\end{aligned} \tag{7}$$

where $[PA]^S = (PA + A^TP)/2$. On the other hand, in view of (6) and Lemma 2.1, $\forall\, t \in [0, +\infty)$ we have

$$\begin{aligned}E(t) = \sum_{i=1}^n p_i \int_0^{z_i(t)} f_i(s) ds &\ge \sum_{i=1}^n \frac{p_i}{2\bar{\ell}_i} f_i^2(z_i(t)) \\ &\ge \xi \sum_{i=1}^n f_i^2(z_i(t)) = \xi\|F(z(t))\|^2\end{aligned} \tag{8}$$

that is, $\|F(z(t))\| \le \sqrt{E(t)/\xi}$ where $\xi = \min_{1\le i\le n}\{p_i/(2\bar{\ell}_i)\}$. Then, from (7) we get

$$\frac{dE(t)}{dt} \le p_{\max}\|F(z)\|\, \|\tilde{u}(t)\| \le p_{\max}\sqrt{\frac{E(t)}{\xi}}\, \|\tilde{u}(t)\|$$

that is

$$\frac{dE(t)}{\sqrt{E(t)}} \le \frac{p_{\max}}{\sqrt{\xi}} \|\tilde{u}(t)\| dt.$$

Taking integral on both sides yields

$$\begin{aligned}2\left(\sqrt{E(t)} - \sqrt{E(0)}\right) &\le \frac{p_{\max}}{\sqrt{\xi}} \int_0^t \|\tilde{u}(s)\| ds \\ &\le \frac{p_{\max}}{\sqrt{\xi}} \int_0^{+\infty} \|\tilde{u}(s)\| ds\end{aligned}$$

that is

$$E(t) \le \left[\sqrt{E(0)} + \frac{p_{\max}}{2\sqrt{\xi}} \int_0^{+\infty} \|\tilde{u}(s)\| ds\right]^2 \tag{9}$$

which is bounded. As a result, $\|F(z(t))\|$ is bounded for $t \in [0, +\infty)$ according to (8). This implies that $g_i(x_i(t))$, $i = 1, 2, \ldots, n$, are all bounded.

Now we arbitrarily choose a sequence $\{t_n\}$ such that $t_n \to +\infty$ as $n \to +\infty$. For $\{x_i(t_n)\}$, three cases are concerned with as follows:

i) $\{x_i(t_n)\}$ is bounded. There must exist a subsequence $\{t_{n_k}\} \subseteq \{t_n\}$ such that

$$\lim_{k\to+\infty} x_i(t_{n_k}) = \bar{x}_i^* \text{ and } \lim_{k\to+\infty} g_i(x_i(t_{n_k})) = g_i(\bar{x}_i^*) \quad (10)$$

where $g_i(\bar{x}_i^*)$ may be different from $g_i(x_i^*)$.

ii) $\{x_i(t_n)\}$ is unbounded and there exists a subsequence $\{t_{n_k}\} \subseteq \{t_n\}$ such that $\lim_{k\to+\infty} x_i(t_{n_k}) \to +\infty$. From (9), we have that $E(t)$ in (6) is bounded on $t \in [0, +\infty)$. Then, one can see that

$$\lim_{k\to+\infty} \int_0^{z_i(t_{n_k})} f_i(s)ds = \lim_{k\to+\infty} \int_0^{x_i(t_{n_k})-x_i^*} f_i(s)ds = \int_0^{+\infty} f_i(s)ds$$

is bounded. This implies that $\lim_{s\to+\infty} f_i(s) = 0$. Since $f_i(0) = 0$ and $f_i(\theta)$ is monotone nondecreasing function on R, we see that $f_i(\theta) \equiv 0, \forall\, \theta \in [0, +\infty)$. As a result, $f_i(z_i) = g_i(x_i) - g_i(x_i^*) = 0, \forall\, x_i \geq x_i^*$. Let $\bar{x}_i^* = x_i^*$. We have

$$\lim_{k\to+\infty} g_i(x_i(t_{n_k})) = g_i(\bar{x}_i^*).$$

iii) $\{x_i(t_n)\}$ is unbounded and there exists a subsequence $\{t_{n_k}\} \subseteq \{t_n\}$ such that $\lim_{k\to+\infty} x_i(t_{n_k}) \to -\infty$. Based on boundedness of $E(t)$ in (6), we know that

$$\lim_{k\to+\infty} \int_0^{z_i(t_{n_k})} f_i(s)ds = \lim_{k\to+\infty} \int_0^{x_i(t_{n_k})-x_i^*} f_i(s)ds = \int_0^{-\infty} f_i(s)ds$$

is bounded. This implies that $\lim_{s\to-\infty} f_i(s) = 0$. Since $f_i(0) = 0$ and $f_i(\theta)$ is monotone nondecreasing function on R, we see that $f_i(\theta) \equiv 0, \forall\, \theta \in (-\infty, 0]$. As a result, $f_i(z_i) = g_i(x_i) - g_i(x_i^*) = 0, \forall\, x_i \leq x_i^*$. Let $\bar{x}_i^* = x_i^*$. We have

$$\lim_{k\to+\infty} g_i(x_i(t_{n_k})) = g_i(\bar{x}_i^*).$$

Based on the three cases above, we can conclude that there exist $\bar{x}_i^*$ and a subsequence $\{t_{n_k}\} \subseteq \{t_n\}$ such that

$$\lim_{k\to+\infty} g_i(x_i(t_{n_k})) = g_i(\bar{x}_i^*), \quad i = 1, 2, \ldots, n$$

where $g_i(\bar{x}_i^*)$ may be different from $g_i(x_i^*)$.

In the following, let $\tilde{u}(t) = u(t) - \overline{u}$ and $\tilde{z} = (\tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n)^T = x - \bar{x}^*$. Then the RNN model (1) can be transformed into the following equivalent system

$$\frac{d\tilde{z}}{dt} = W\tilde{F}(\tilde{z}) + \tilde{u}(t), \quad \tilde{z}(0) = \tilde{z}_0 = x_0 - \bar{x}^* \quad (11)$$

where $\tilde{F}(\tilde{z}) = (\tilde{f}_1(\tilde{z}_1), \ldots, \tilde{f}_n(\tilde{z}_n))^T = g(\tilde{z} + \bar{x}^*) - g(\bar{x}^*)$ is a globally Lipschitz continuous and monotone nondecreasing activation function and $\tilde{F}(0) = 0$. Similar to the proof of boundedness of $\|F(z(t))\|$, we can see that $\|\tilde{F}(\tilde{z}(t))\|$ is bounded, that is, there exists a positive number M such that $\|\tilde{F}(\tilde{z}(t))\| \leq M$, $\forall\, t \geq 0$. Note that W is Lyapunov diagonally semistable; that is, there exists a positive definite matrix $P = \mathrm{diag}(p_1, \ldots, p_n)$ such that $PW + W^TP \leq 0$. Let $v(t) = (v_1(t), \ldots, v_n(t))^T \triangleq (\tilde{f}_1(\tilde{z}_1(t)), \ldots, \tilde{f}_n(\tilde{z}_n(t)))^T$. Now we consider the following differentiable function

$$E(v(t), t) = \sum_{i=1}^{n} p_i \int_0^{\tilde{z}_i(t)} \tilde{f}_i(s)ds + \eta \exp(-\beta_{\min} t) \quad (12)$$

where

$$\eta > p_{\max} M \frac{\sqrt{n}\alpha_{\max}}{\beta_{\min}}. \quad (13)$$

From (8) it follows that

$$\sum_{i=1}^{n} p_i \int_0^{\tilde{z}_i(t)} \tilde{f}_i(s)ds \geq \xi\|v(t)\|^2$$

where $\xi > 0$ is defined in (8). Then

$$E(v(t), t) \geq \xi\|v(t)\|^2 + \eta \exp(-\beta_{\min} t). \quad (14)$$

Computing the time derivative of $E(v(t), t)$ along the trajectory $\tilde{z}(t)$ of system (11) yields

$$\begin{aligned}\frac{dE(v(t), t)}{dt} &= \tilde{F}(\tilde{z})^T P(W\tilde{F}(\tilde{z}) + \tilde{u}(t)) - \eta\beta_{\min} \exp(-\beta_{\min} t)\\ &= F(\tilde{z})^T [PW]^S \tilde{F}(\tilde{z}) + \tilde{F}(\tilde{z})^T P\tilde{u}(t)\\ &\quad - \eta\beta_{\min} \exp(-\beta_{\min} t)\\ &\leq p_{\max}\|\tilde{F}(\tilde{z})\|\, \|\tilde{u}(t)\| - \eta\beta_{\min} \exp(-\beta_{\min} t)\\ &\leq p_{\max} M\sqrt{n}\alpha_{\max} \exp(-\beta_{\min} t)\\ &\quad - \eta\beta_{\min} \exp(-\beta_{\min} t)\\ &= [p_{\max} M\sqrt{n}\alpha_{\max} - \eta\beta_{\min}]\\ &\quad \times \exp(-\beta_{\min} t) < 0, \quad \forall\, t \in [0, +\infty) \end{aligned} \quad (15)$$

where η is given in (13) and $[PW]^S = (PW + W^TP)/2$. Thus, $E(v(t), t)$ is a strictly monotone decreasing function on $t \in [0, +\infty)$ and is bounded. As a result, there exists a number $E^* \geq 0$ such that

$$\lim_{t\to+\infty} E(v(t), t) = E^*. \quad (16)$$

Next we will show that $E^* = 0$. To do so, we consider the following three cases for $\{x_i(t_{n_k})\}$ corresponding to cases i)–iii) above, respectively.

1) $\lim_{k\to+\infty} x_i(t_{n_k}) = \bar{x}_i^*$. Noticing the nondecreasing monotonicity of $\tilde{f}_i(\theta)$ with respect to $\theta \in R$ and boundedness of $\tilde{f}_i(\tilde{z}_i(t))$, we have

$$\begin{aligned}0 &\leq \lim_{k\to+\infty} \int_0^{\tilde{z}_i(t_{n_k})} \tilde{f}_i(s)ds \leq \lim_{k\to+\infty} \tilde{z}_i(t_{n_k})\tilde{f}_i(\tilde{z}_i(t_{n_k}))\\ &= \lim_{k\to+\infty} (x_i(t_{n_k}) - \bar{x}_i^*)\tilde{f}_i(\tilde{z}_i(t_{n_k})) = 0\end{aligned}$$

that is

$$\lim_{k\to+\infty} \int_0^{\tilde{z}_i(t_{n_k})} \tilde{f}_i(s)ds = 0.$$

2) $\lim_{k\to+\infty} x_i(t_{n_k}) \to +\infty$. Noticing the boundedness of $E(v(t), t)$ on $t \in [0, +\infty)$, similar to ii) above we can

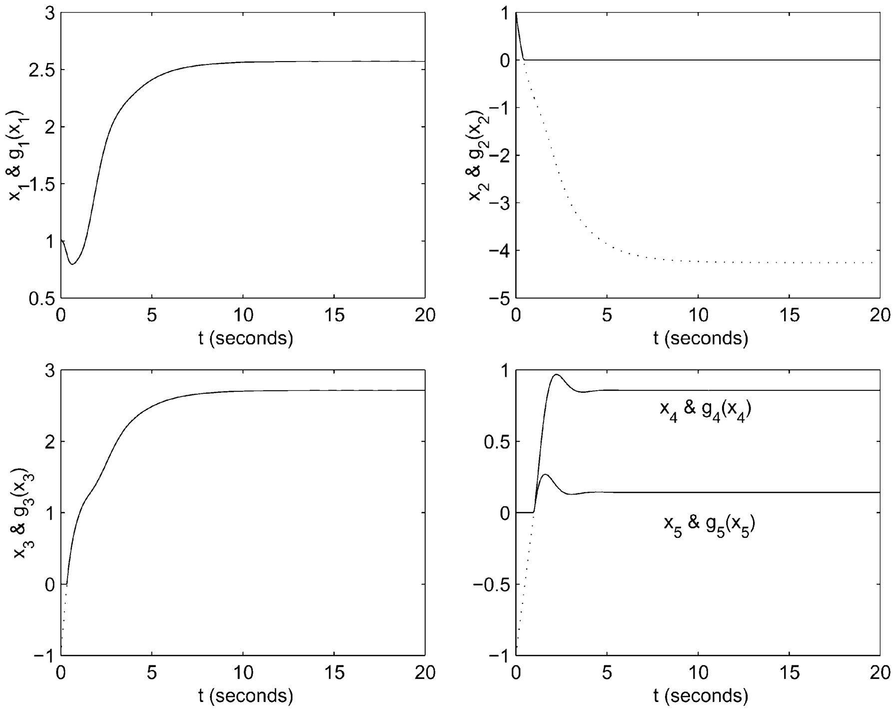

Fig. 1. Output convergence of the network in the example with $x_0 = (1, 1, -1, -1, -1)^T$.

derive that $\tilde{f}_i(\tilde{z}_i) = g_i(x_i) - g_i(\bar{x}_i^*) = 0, \forall\, x_i \geq \bar{x}_i^*$ (i.e., $\forall\, \tilde{z}_i \geq 0$) and

$$\lim_{k\to+\infty}\int_0^{\tilde{z}_i(t_{n_k})}\tilde{f}_i(s)ds = \lim_{k\to+\infty}\int_0^{x_i(t_{n_k})-\bar{x}_i^*}\tilde{f}_i(s)ds = \int_0^{+\infty}0ds = 0.$$

3) $\lim_{k\to+\infty} x_i(t_{n_k}) \to -\infty$. Noticing the boundedness of $E(v(t),t)$ on $t \in [0,+\infty)$, similar to iii) above we can derive that $\tilde{f}_i(\tilde{z}_i) = g_i(x_i) - g_i(\bar{x}_i^*) = 0, \forall\, x_i \leq \bar{x}_i^*$ (i.e., $\forall\, \tilde{z}_i \leq 0$) and

$$\lim_{k\to+\infty}\int_0^{\tilde{z}_i(t_{n_k})}\tilde{f}_i(s)ds = \lim_{k\to+\infty}\int_0^{x_i(t_{n_k})-\bar{x}_i^*}\tilde{f}_i(s)ds = \int_0^{-\infty}0ds = 0.$$

In view of 1)–3) above, we have

$$\lim_{k\to+\infty}\int_0^{\tilde{z}_i(t_{n_k})}\tilde{f}_i(s)ds = 0.$$

Therefore

$$\lim_{k\to+\infty}E(v(t_{n_k}),t_{n_k}) = \lim_{k\to+\infty}\sum_{i=1}^{n}p_i\int_0^{\tilde{z}_i(t_{n_k})}\tilde{f}_i(s)ds + \lim_{k\to+\infty}\eta\exp(-\beta_{\min}t_{n_k}) = 0.$$

Then, from (16) it follows that

$$E^* = \lim_{t\to+\infty}E(v(t),t) = \lim_{k\to+\infty}E(v(t_{n_k}),t_{n_k}) = 0.$$

Thus, in view of (14) we have

$$0 = \lim_{t\to+\infty}E(v(t),t) \geq \xi\lim_{t\to+\infty}\|v(t)\|^2 + \lim_{t\to+\infty}\eta\exp(-\beta_{\min}t) = \xi\lim_{t\to+\infty}\|v(t)\|^2$$

which leads to $\lim_{t\to+\infty}\|v(t)\|^2 \leq 0$ and consequently, $\lim_{t\to+\infty}\|v(t)\|^2 = 0$ and $\lim_{t\to+\infty}v(t) = 0$; that is, $\lim_{t\to+\infty}\tilde{F}(\tilde{z}(t)) = 0$. This implies that $\lim_{t\to+\infty}g(x(t)) = g(\bar{x}^*)$. Based on (11) and $\lim_{t\to+\infty}\tilde{u}(t) = 0$, it follows that $\lim_{t\to+\infty}d\tilde{z}/dt = 0$. Since $\tilde{z} = x - \bar{x}^*$, one can see that $\lim_{t\to+\infty}dx/dt = 0$. From (1) it follows that

$$0 = \lim_{t\to+\infty}(Wg(x(t)) + u(t)) = \lim_{t\to+\infty}Wg(x(t)) + \lim_{t\to+\infty}u(t) = Wg(\bar{x}^*) + u.$$

This completes the proof of the theorem.

Remark 3.2: In Theorem 3.1, one needs to check the condition that there exists a vector $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T \in R^n$ such that $Wg(x^*)+\overline{u} = 0$. By Lemma 2.2, this condition is satisfied if V in (3) is not an empty set. In Theorem 3.1 the vector $g(x)$ satisfying $Wg(x) + \overline{u} = 0$ may not be unique when W is singular.

Remark 3.3: The RNN model (1) with constant thresholds is a special case of the general CAM network (13) in [5]. To guarantee that every trajectory approaches one of equilibrium points, the symmetry condition (15) in [5] is required. However, the weight matrix W in Theorem 3.1 may not be symmetric.

Remark 3.4: Some specific recurrent neural networks with time-varying thresholds have been studied in [17], [18], [20]–[22]. On one hand, the time-varying thresholds in [17], [18], [20]–[22] satisfy (5). On the other hand, one can easily check that the connection weight matrices W of the networks

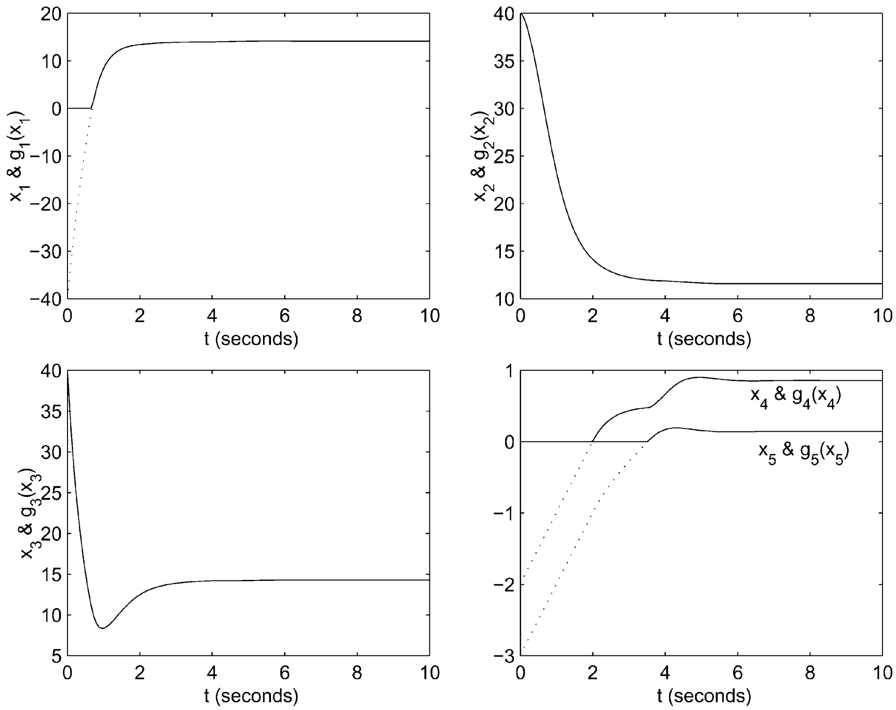

Fig. 2. Output convergence of the network in the example with $x_0 = (-40, 40, 40, -2, -3)^T$.

in [17], [18], [20]–[22] satisfy $W + W^T \leq 0$. As a result, Theorem 3.1 extends the results in [17], [18], [20]–[22] to more general cases as far as the connection weight matrix is concerned.

IV. An Illustrative Example

Consider the RNN model (1) where

$$W = \begin{bmatrix} -2 & 1 & 1 & 2 & -2 \\ 0 & -1 & 1 & -1 & 1 \\ 2 & 0 & -2 & -1 & 1 \\ 0 & 0 & 0 & -2 & 5 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

$$u(t) = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \exp(-100t) \\ \exp(-100t) \\ \exp(-100t) \\ \exp(-100t) \\ \exp(-100t) \end{bmatrix}$$

and $g_i(x_i) = \max\{0, x_i\}$, $i = 1, 2, 3, 4, 5$. It is easy to see that W is singular and is not symmetric. Letting $P = \mathrm{diag}(1, 1, 1, 1, 5)$, one can verify that $PW + W^T P$ is negative semidefinite; that is, W is Lyapunov diagonally semistable. On the other hand, letting the initial condition $x_0 = (1, 1, -1, -1, -1)^T$, Fig. 1 shows that the positive half trajectory $x(t)$ of the RNN model (1) converges to the point $x^* = (2.5714, -4.2576, 2.7143, 0.8571, 0.1429)^T$ and output $g(x)$ of the RNN model (1) converges to the point $g(x^*) = (2.5714, 0, 2.7143, 0.8571, 0.1429)^T$ where $g(x^*)$ satisfies $Wg(x^*) + \overline{u} = 0$ with $\overline{u} = (1, -2, 1, 1, 1)^T$. Since the conditions of Theorem 3.1 are satisfied, this network is globally output convergent. To verify this, we choose a different initial point $x_0 = (-40, 40, 40, -2, -3)^T$. Fig. 2 shows that positive half trajectory $x(t)$ and output $g(x)$ of the RNN model (1) all converge the point $\bar{x}^* = (14.1541, 11.5827, 14.2970, 0.8571, 0.1429)^T$ and moreover $Wg(\bar{x}^*) + \overline{u} = 0$. Obviously, $g(\bar{x}^*) \neq g(x^*)$. In Figs. 1 and 2, the dotted lines correspond to $x_i(t)$ and the solid lines correspond to $g_i(x_i(t))$, $i = 1, 2, 3, 4, 5$.

V. Conclusions

In this paper, we have established global output convergence for a class of continuous-time recurrent neural networks with globally Lipschitz continuous and monotone nondecreasing activation functions and locally Lipschitz continuous time-varying thresholds. One sufficient condition to guarantee the global output convergence of this class of neural networks has been established. This result extends existing results and is very useful in the design of recurrent neural networks with time-varying thresholds.

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