side of the lines $0.5x_1 + x_2 + 1 = 0$ and $x_1 + 0.5x_2 + 1 = 0$ fully belong to the basin of attraction of $E1$. The above lines correspond to $dx_1/dt = 0$ and $dx_2/dt = 0$ (in region 5), and for the points above these lines $dx_1/dt > 0$ and $dx_2/dt > 0$, i.e., the state trajectories are directed toward the region 5. The separation lines in regions 8 and 4 are $x_1 = 0$ and $x_2 = 0$, respectively, because they correspond to $dx_1/dt = 0$ and $dx_2/dt = 0$, and the state trajectories are directed toward the regions 8 and 4. This set of lines is drawn in Fig. 1 in bold.

Now we will search separation lines for the parts that belong to the basin of attraction of $E2$. Reasoning similar to that above leads to the conclusion that in region 5 the same two lines ($0.5x_1 + x_2 + 1 = 0$ and $x_1 + 0.5x_2 + 1 = 0$) can be used. However, now all points below the two lines will fully belong to the basin of attraction of $E2$ because the state trajectories are directed toward region 5. The separation line in regions 8 is $x_1 = -0.5$ because in the left part of this line the state trajectories are directed toward region 8. Similarly, due to the symmetry, the separation line for the region 4 is $x_2 = -0.5$.

From the above considerations, it has become apparent that there are still subregions which can belong to each of the basin of attractions. This part can be reduced using more separation lines. However, the general approach would be to search for some high-order manifold for the separation. However, so far it is not known how to find or construct this manifold. The use of hyperplanes (lines), as in the example, will only give a first-order estimation of the basin of attraction.

V. Conclusions

In this brief we suggest a method for determining an estimation of the basin of attractions for the stable equilibrium points in Cellular Neural Networks. It is valid also for continuous time Hopfield networks if the output functions are piecewise instead of sigmoid functions. The method is based on determining the so-called tree of regions connected with each stable equilibrium point, and gives more insight into how the basin of attractions are situated. The method is also useful when it is necessary to determine where (on which stable equilibrium point) a trajectory, started from a given (binary) initial condition, will settle after the transient decays, e.g., which basin of attraction this (binary) initial condition belongs to.

It should be stressed that the problem of the estimation of the basin of attractions is extremely complex and our approach also becomes complex for high-dimension CNN's. Nevertheless, the calculations for determining the sign of dx_i/dt on the separation hyperplanes are not time consuming; the first-order estimation can simply be automated. The problem arises with the number of the regions which belong to several basin of attractions when the order of CNN increases. The approach in Section IV should be applied for each of these regions, and the improvement is an open question for further research.

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Lyapunov Stability of Two-Dimensional Digital Filters with Overflow Nonlinearities

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***Abstract*—In this short paper, the second method of Lyapunov is utilized to establish sufficient conditions for the global asymptotic stability of the trivial solution of zero-input two-dimensional (2-D) Fornasini–Marchesini state-space digital filters which are endowed with a general class of overflow nonlinearities. Results for the global asymptotic stability of the null solution of the 2-D Fornasini–Marchesini second model with overflow nonlinearities are established. Several classes of Lyapunov functions are used in establishing the present results, including vector norms and the quadratic form. When the quadratic form Lyapunov functions are considered, the present results involve necessary and sufficient conditions under which positive definite matrices can be used to generate Lyapunov functions for 2-D digital filters with overflow nonlinearities.**

***Index Terms*—Finite wordlength effects, Lyapunov methods, 2-D systems.**

I. Introduction

Consider zero-input two-dimensional (2-D) nonlinear digital filters described by

$$x(k+1, l+1) = f[A_1 x(k, l+1) + A_2 x(k+1, l)], \quad k \geq 0, \ l \geq 0 \tag{1}$$

where $x \in R^n$, $A_1 \in R^{n \times n}$, $A_2 \in R^{n \times n}$, and $f(\cdot)$ represents overflow nonlinearities. For system (1), assume a finite set of initial conditions, i.e., assume that there exist two positive integers K and L, such that

$$x(k, 0) = 0 \quad \text{for } k \geq K \quad \text{and } x(0, l) = 0 \quad \text{for } l \geq L. \tag{2}$$

System (1) is usually referred to as the zero-input Fornasini–Marchesini second model [10] implemented in finite word-length format.

In the *implementation* of digital filters, signals are usually represented and processed in a finite word-length format which gives rise to several kinds of nonlinear effects, such as overflow and quantization. Since finite word-length realizations of digital filters result in systems which inherently are *nonlinear*, the asymptotic stability of such filters (under zero-input) is of great interest in practice (cf., [1]–[9], [11]–[15], [17]–[21]). The global asymptotic

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stability of the null solution guarantees the nonexistence of limit cycles (overflow oscillations) in the realized digital filters. In the present paper, results for the global asymptotic stability of 2-D digital filters with overflow nonlinearities will be established. Quantization effects will not be considered.

Associated with the nonlinear digital filter described by (1), consider linear digital filters given by

$$w(k+1, l+1) = A_1 w(k, l+1) + A_2 w(k+1, l), \quad k \geq 0, \, l \geq 0 \tag{3}$$

where A_1 and A_2 are defined in (1), and $w \in R^n$. A finite set of initial conditions as in (2) will be assumed for system (3). For 2-D linear digital filters described by (3), i.e., filters (1) implemented in an ideal format, a result proved in [11] and [12] states that the 2-D digital filter (3) is asymptotically stable, if there exists a positive definite matrix $P \in R^{n \times n}$, such that

$$Q = \begin{bmatrix} \alpha P & \vdots & 0 \\ \cdots & \vdots & \cdots \\ 0 & \vdots & \beta P \end{bmatrix} - A^T P A \tag{4}$$

is positive definite, where $A = [A_1 \vdots A_2]$, $\alpha, \beta > 0$ and $\alpha + \beta = 1$. This result has been further improved in [16].

In this short paper, the asymptotic stability of the 2-D digital filter (1) with initial conditions (2) will be studied. In particular, Lyapunov's Second Method will be utilized to establish results for the global asymptotic stability of the null solution of 2-D systems described by (1). Results of the present paper for *nonlinear* digital filters (1) constitute modifications of the results in [11] and [16] for *linear* digital filters (3). For example, it will be shown that with certain constraints on the matrix P, condition (4) can also be applied to system (1). Several classes of Lyapunov functions will be used in establishing the present results including the quadratic form. When the quadratic form Lyapunov functions are used, the present results involve necessary and sufficient conditions under which positive definite matrices can be used to generate Lyapunov functions for the systems considered herein.

II. Main Results

In the present paper, the nonlinearity $f: R^n \to R^n$ representing overflow effects in 2-D digital filters (1) is defined by

$$f(x) = [\varphi(x_1), \cdots, \varphi(x_N)]^T. \tag{5}$$

The function $\varphi: R \to [-1, 1]$ in (5) is piecewise continuous and is defined by (see Fig. 1)

$$\begin{cases} L \leq \varphi(x_i) \leq L_1, & x_i > 1 \\ \varphi(x_i) = x_i, & -1 \leq x_i \leq 1 \\ -L_2 \leq \varphi(x_i) \leq -L, & x_i < -1 \end{cases} \tag{6}$$

where $-1 \leq L \leq L_1, L_2 \leq 1$. For given L, L_1, and L_2, when $|x_i| > 1$, $\varphi(x_i)$ in (6) may assume *any* value in the crosshatched region in Fig. 1 including $\pm L$, L_1, and $-L_2$. The class of overflow nonlinearities considered herein will be called *generalized overflow characteristics*. These nonlinearities constitute a generalization of the usual types of overflow arithmetic employed in practice such as *zeroing* ($L = L_1 = L_2 = 0$), *two's complement* ($L = -1$, $L_1 = L_2 = 1$), *triangular* ($L = -1$, $L_1 = L_2 = 1$), and *saturation* ($L = L_1 = L_2 = 1$) overflow characteristics. In the present approach, these nonlinearities, although satisfying sector conditions (cf., [3], [4], [9], [11]), are here characterized by the range of the nonlinear function representing the overflow arithmetic.

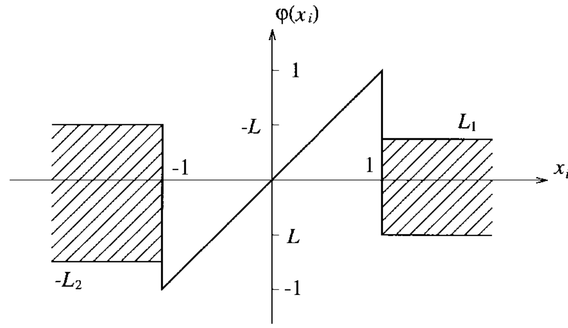

Fig. 1. The generalized overflow nonlinearity described by (7).

In analyzing the stability of the equilibrium $x_e = 0$ of the 2-D system (1), a class of Lyapunov functions for the linear system (3) will be used. Specifically, the following assumption is made.

Assumption (A-1): Assume that for system (3) there exists a positive definite and radially unbounded function $V: R^n \to R$ with the following properties

i) Denote

$$\zeta(k, l) = \begin{bmatrix} w(k, l+1) \\ w(k+1, l) \end{bmatrix}.$$

The function $DV_{(3)}: R^{2n} \to R$, defined by

$$\begin{aligned} DV_{(3)}[\zeta(k, l)] &\triangleq V[w(k+1, l+1)] - \alpha V[w(k, l+1)] \\ &\quad - \beta V[w(k+1, l)] \\ &= V[A_1 w(k, l+1) + A_2 w(k+1, l)] \\ &\quad - \alpha V[w(k, l+1)] - \beta V[w(k+1, l)] \end{aligned} \tag{7}$$

is negative definite for all $\zeta(k, l) \in R^{2n}$, where $\alpha, \beta > 0$ and $\alpha + \beta = 1$. [Note that the right hand side of (7) is indeed a function of $\zeta(k, l)$].

ii) For all $w \in R^n$, it is true that

$$V[f(w)] \leq V(w) \tag{8}$$

where f represents the overflow nonlinearity defined by (5) and (6). ■

The following result can now be established.

Theorem 1: If Assumption (A-1) holds, the equilibrium $x_e = 0$ of the 2-D system (1) is globally asymptotically stable.

Proof: Since (A-1) is true, there exists a positive definite and radially unbounded function V for system (3), such that (8) is true, which in turn implies that

$$\begin{aligned} &V\{f[A_1 w(k, l+1) + A_2 w(k+1, l)]\} \\ &\quad \leq V[A_1 w(k, l+1) + A_2 w(k+1, l)] \end{aligned}$$

for all

$$\begin{bmatrix} w(k, l+1) \\ w(k+1, l) \end{bmatrix} \in R^{2n}.$$

Also, by (7),

$$\begin{aligned} V[w(k+1, l+1)] &= V[A_1 w(k, l+1) + A_2 w(k+1, l)] \\ &< \alpha V[w(k, l+1)] + \beta V[w(k+1, l)] \end{aligned}$$

for all $[w(k, l+1)^T, \ w(k+1, l)^T] \neq 0$. Thus, for the 2-D system (1), it is clear that

$$\begin{aligned} V[x(k+1, l+1)] &= V\{f[A_1 x(k, l+1) + A_2 x(k+1, l)]\} \\ &< \alpha V[x(k, l+1)] + \beta V[x(k+1, l)] \end{aligned} \tag{9}$$

for all $[x(k, l+1)^T, x(k+1, l)^T] \neq 0$.

Let $D(d)$ denote the set defined by

$$D(d) \triangleq \{(k, l): k + l = d, \qquad k \geq 0, l \geq 0\}$$

for some integer $d > 0$. [$D(d)$ represents indices along the diagonal.] For any integer $d \geq \max\{K, L\}$, compute from (9) and $\beta + \alpha = 1$,

$$\begin{aligned} &\sum_{(k, l) \in D(d)} V[x(k, l)] \\ &= V[x(0, d)] + V[x(1, d-1)] + V[x(2, d-2)] + \cdots \\ &\quad + V[x(d-1, 1)] + V[x(d, 0)] \\ &= \alpha V[x(0, d)] + \beta V[x(1, d-1)] \\ &\quad + \alpha V[x(1, d-1)] + \beta V[x(2, d-2)] \\ &\quad + \alpha V[x(2, d-2)] + \cdots + \beta V[x(d-1, 1)] \\ &\quad + \alpha V[x(d-1, 1)] + \beta V[x(d, 0)] \\ &> V[x(1, d)] + V[x(2, d-1)] + \cdots + V[x(d, 1)] \\ &= \sum_{(k, l) \in D(d+1)} V[x(k, l)]. \end{aligned} \tag{10}$$

In the above, the fact that

$$x(0, d) = x(d, 0) = x(0, d+1) = x(d+1, 0) = 0$$

and the positive definiteness of the function V have been used.

The rest of the proof follows along similar steps as in the proofs of Lemma A.2 and Theorem 1 in [15]; i.e., it can be proved that the equilibrium $x_e = 0$ of the 2-D system (1) is stable, and that for any initial conditions satisfying (2),

$$\lim_{k \to \infty \text{ and/or } l \to \infty} x(k, l) = \lim_{k+l \to \infty} x(k, l) = 0.$$

Therefore, the equilibrium $x_e = 0$ of the 2-D system (1) is globally asymptotically stable. ■

Remark 1: The condition $\alpha + \beta = 1$ in Theorem 1 can be replaced by $\alpha + \beta \leq 1$, which will not affect the validity of the theorem. ■

A function V satisfying Theorem 1 will be referred to as a *Lyapunov function* for the 2-D system (1).

In particular, when the function V is chosen as $V(w) = \|w\|_v$, where $\|w\|_v$ denotes any vector norm satisfying that

$$\|f(w)\|_v \leq \|w\|_v, \qquad \text{for all } w \in R^n \tag{11}$$

for f defined by (5) and (6), the following result can be established.

Corollary 1: The equilibrium $x_e = 0$ of the 2-D system (1) is globally asymptotically stable if

$$\|A_1\|_v + \|A_2\|_v < 1 \tag{12}$$

where $\|A_i\|_v$, $i = 1, 2$, denotes the matrix norm induced by the vector norm $\|w\|_v$ satisfying (11).

Proof: It suffices to show that if (12) is true, then Assumption (A-1) is satisfied.

Clearly, for any two vectors $x, y \in R^n$,

$$\begin{aligned} V(A_1 x + A_2 y) &= \|A_1 x + A_2 y\|_v \leq \|A_1 x\|_v + \|A_2 y\|_v \\ &\leq \|A_1\|_v \|x\|_v + \|A_2\|_v \|y\|_v. \end{aligned}$$

Since $\|A_1\|_v + \|A_2\|_v < 1$, there exists an $\varepsilon > 0$ such that $\|A_1\|_v + \|A_2\|_v = 1 - \varepsilon$. Choosing $\alpha = \|A_1\|_v + \varepsilon/2$ and $\beta = \|A_2\|_v + \varepsilon/2$, one can see that $\alpha + \beta = 1$, $\alpha > \|A_1\|_v$, and $\beta > \|A_2\|_v$. Hence,

$$V(A_1 x + A_2 y) < \alpha \|x\|_v + \beta \|y\|_v = \alpha V(x) + \beta V(y) \tag{13}$$

for all $[x^T, y^T] \neq 0$, i.e., $V(w) = \|w\|_v$ satisfies condition (7). In view of (11), it is clear now that Assumption (A-1) is satisfied. ■

Remark 2: One of the examples of vector norms which satisfies (11) is the l_p vector norm given by

$$\|w\|_p = \left(\sum_{i=1}^{n} |w_i|^p \right)^{1/p}$$

where $1 \leq p \leq \infty$. ■

In the following, a quadratic form Lyapunov function for system (1) will be considered. In deriving the next result, the following assumption which characterizes a class of positive definite matrices will be made. (Throughout, the term *positive definite matrix* stands for a *symmetric* matrix with positive eigenvalues.)

Assumption (A-2): Let f be defined by (5) and (6). Assume that there exists a positive definite matrix $H \in R^{n \times n}$ such that

$$f(x)^T H f(x) < x^T H x$$

for all $x \in R^n, \ x \notin D^n \triangleq \{x \in R^n: -1 \leq x_i \leq 1\}$. ■

The next result provides a *necessary and sufficient* condition for matrices to satisfy Assumption (A-2) when f represents the generalized overflow arithmetic. This result is very useful in applications.

Lemma 1: Assume that f is defined by (5) and (6). An $n \times n$ positive definite matrix $H = [h_{ij}]$ satisfies Assumption (A-2) *if and only if*

$$(1 + L) h_{ii} \geq 2 \sum_{j=1, j \neq i}^{n} |h_{ij}|, \qquad i = 1, \cdots, n. \tag{14}$$

Proof: This lemma can be proved following steps similar to those in [14]. The difference between the present form and the result in [14] is the choice for L_1 and L_2. In [14], it is chosen that $L_1 = 1$ and $L_2 = 1$; while in the present case, it is only required that $L \leq L_1, L_2 \leq 1$. ■

The following result can now be established based on Theorem 1.

Corollary 2: The equilibrium $x_e = 0$ of the 2-D digital filter (1) is globally asymptotically stable, if there exists a positive definite matrix $H \in R^{n \times n}$ satisfying Assumption (A-2), such that

$$Q = \begin{bmatrix} \alpha H & \vdots & 0 \\ \cdots & \vdots & \cdots \\ 0 & \vdots & \beta H \end{bmatrix} - A^T H A \tag{15}$$

is positive definite, where $A = [A_1 \vdots A_2]$, $\alpha, \beta > 0$, and $\alpha + \beta = 1$.

Proof: For system (1), choose the positive definite and radially unbounded Lyapunov function $V(x) = x^T H x$. Since H satisfies Assumption (A-2), one can see that

$$V[f(x)] = f(x)^T H f(x) \leq x^T H x = V(x)$$

for all $x \in R^n$. Thus, for system (1)

$$\begin{aligned} &V[x(k+1, l+1)] \\ &= V\{f[A_1 x(k, l+1) + A_2 x(k+1, l)]\} \\ &\leq V[A_1 x(k, l+1) + A_2 x(k+1, l)] \\ &= [x(k, l+1)^T \ x(k+1, l)^T] A^T H A \begin{bmatrix} x(k, l+1) \\ x(k+1, l) \end{bmatrix} \end{aligned}$$

for all $x \in R^n$. Using (15), it can be found that

$$\begin{aligned} V[x(k+1, l+1)] &\le \alpha x(k, l+1)^T H x(k, l+1) + \beta x(k+1, l)^T \\ &\cdot H x(k+1, l) - [x(k, l+1)^T \;\; x(k+1, l)^T] \\ &\cdot Q \begin{bmatrix} x(k, l+1) \\ x(k+1, l) \end{bmatrix}. \end{aligned}$$

Therefore,

$$V[x(k+1, l+1)] < \alpha x(k, l+1)^T H x(k, l+1) + \beta x(k+1, l)^T H x(k+1, l)$$

i.e.,

$$V[x(k+1, l+1)] < \alpha V[x(k, l+1)] + \beta V[x(k+1, l)]$$

for all $[x(k, l+1)^T, \;\; x(k+1, l)^T] \neq 0$, since Q is positive definite.

The rest of the proof follows along similar lines as in the proof of Theorem 1. ■

Remark 3: Corollary 2 constitutes a modification of one of the results in [11] [cf., condition (4)]. Specifically, it has been shown in the present paper that the result in [11] can also serve as a criterion for testing the global asymptotic stability of the null solution of digital filter (1) if the matrix P in (4) satisfies the Assumption (A-2). It should also be noted that, for 2-D digital filters with overflow nonlinearities characterized by *sector conditions*, a result is obtained in Theorem 2 of [11] (which requires that the positive definite matrix P be diagonal). The result in Corollary 2 of the present paper and the result in Theorem 2 of [11] constitute different sufficient conditions. It appears that usage of the parameter L given in (6) to characterize overflow nonlinearities may in some cases be more desirable than usage of sector conditions. ■

The following is a special case of Corollary 2 when $H = I$ (the identity matrix) and $\alpha = \beta = 1/2$. This result is very easy to apply.

Corollary 3: The equilibrium $x_e = 0$ of the 2-D digital filter (1) is globally asymptotically stable, if

$$\|A\|_2 = \sqrt{\lambda_M(A^T A)} < \frac{1}{\sqrt{2}}$$

where $A = [A_1 \vdots A_2]$ and

$$\lambda_M\left(A^T A\right) = \max_{1 \le i \le 2n} \lambda_i\left(A^T A\right)$$ ■

The next theorem, for which Corollary 2 can also be considered as a special case, can now be established based on the above results and the results of [16].

Theorem 2: The equilibrium $x_e = 0$ of the 2-D digital filter (1) is globally asymptotically stable, if there exist positive definite matrices H, W_1, and W_2, with H satisfying Assumption (A-2), such that

$$Q = \begin{bmatrix} H^{T/2} W_1 H^{1/2} & \vdots & 0 \\ \cdots\cdots\cdots\cdots & \vdots & \cdots\cdots\cdots\cdots \\ 0 & \vdots & H^{T/2} W_2 H^{1/2} \end{bmatrix} - A^T H A$$

is positive definite and such that $I - W_1 - W_2$ is positive semidefinite, where I denotes the identity matrix, $H = H^{T/2} H^{1/2}$, and $H^{T/2} = (H^{1/2})^T$.

Proof: Choose $V(x) = x^T H x$ and denote

$$\xi(k, l) = \begin{bmatrix} x(k, l+1) \\ x(k+1, l) \end{bmatrix}.$$

By defining

$$\begin{aligned} DV_{(1)}[\xi(k, l)] \triangleq\; & f[A_1 x(k, l+1) + A_2 x(k+1, l)]^T \\ & \cdot H f[A_1 x(k, l+1) + A_2 x(k+1, l)] \\ & - x(k, l+1)^T H^{T/2} W_1 H^{1/2} x(k, l+1) \\ & - x(k+1, l)^T H^{T/2} W_2 H^{1/2} x(k+1, l) \end{aligned}$$

the proof of this theorem follows along steps similar to those in the proof of Theorem 1 and Corollary 2. ■

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