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Source: *Management Science*, Vol. 23, No. 11 (Jul., 1977), pp. 1248-1251

Published by: INFORMS

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Notes*

III

A NOTE ON THE DYNAMIC INVENTORY PROBLEM WITH UNKNOWN DEMAND DISTRIBUTION†

S. PAPACHRISTOS‡

We point out some errors in Iglehart's paper, "The Dynamic Inventory Problem with Unknown Demand Distribution," published in *Management Science* in 1964.

1. Introduction

Iglehart in [1] studied the determination of optimal inventory control policies and the behaviour of the critical levels of these policies in terms of the various parameters involved. Some of his results are given without proof, and it turns out that they are not true. The results in question are: the stochastic order property of the exponential family, Lemma 8; the monotonicity of the sequence $\bar{x}_N(s, n)$ of the critical levels, Theorem 2.a; the convergence of this sequence and of the sequence $\bar{x}(s, n)$, Theorems 3.e and 6.b; and finally the uniform convergence of the sequence $p(\xi/v, n)$, Lemma 2.

In §§2.1 and 2.2 we present counterexamples for the first part of Lemma 8 and Theorem 2.a. In §3 we discuss the validity of Theorems 3.e, 6.b, and Lemma 2, and finally in §4 we suggest some modifications which are felt to be necessary for strengthening some of the results of the paper mentioned above. We use the notation and assumptions of [1].

2. Counterexamples on Lemma 8 and Theorem 2.a

2.1.

Let $\xi_1, \xi_2, \dots, \xi_n$ be the values of n independent observations of a random variable ξ which is distributed exponentially with p.d.f. $\phi(\xi/\omega) = \beta(\omega)e^{-\omega\xi}r(\xi)$, where $\xi \geq 0$, ω is an unknown parameter and $[\beta(\omega)]^{-1} = \int_0^\infty e^{-\omega\xi}r(\xi)d\xi$.

The pair (s, n) or (Q, n) , where $s = Q/n$ and $Q = \sum_{i=1}^n \xi_i$, is a sufficient statistic for the parameter ω (see [5]). If $f(\omega)$ is the p.d.f. of ω , the conditional p.d.f. of ξ given (s, n) is

$$\phi(\xi/s, n) = r(\xi) \int_0^\infty \beta^{n+1}(\omega) e^{-n\omega s} e^{-\xi\omega} f(\omega) d\omega / \int_0^\infty \beta^n(\omega) e^{-n\omega s} f(\omega) d\omega. \quad (1)$$

Let $\Phi(x/s, n)$ be the c.d.f. of the random variable ξ given (s, n) and define

$$D(x/s, n) = \Phi(x/s, n+1) - \Phi(x/s, n). \quad (2)$$

The first part of Lemma 8 of [1] says that

$$\phi(\xi/s, n+1) \subset \phi(\xi/s, n), \quad \text{i.e.,} \quad D(x/s, n) \geq 0 \quad (3)$$

for every $x \geq 0$ and any given s, n .

* All Notes are refereed.

† Accepted by Edward J. Ignall, former Departmental Editor for Dynamic Programming and Inventory Theory; received June 1976. This paper has been with the author 1 month, for 1 revision.

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We shall illustrate that this part of the lemma is not true. Let us take

$$\phi(\xi/\omega) = (1/\omega)e^{-\xi/\omega}, \quad (4)$$

$$f(\omega) = (1/2)e^{-\nu/\omega}(\nu/\omega)^4(1/\nu), \quad (5)$$

with $\xi \geq 0$, $\omega > 0$ and ν a known parameter. The density $\phi(\xi/\omega)$ belongs to the exponential family as Lemma 8 requires and $f(\omega)$ is a member of the Inverted Gamma-1 family of densities (see [5]).

With $\phi(\xi/\omega)$, $f(\omega)$ defined by (4), (5), the expression for $\phi(\xi/s, n)$ in (1) becomes

$$\phi(\xi/s, n) = (n+3)(\nu + ns)^{n+3}/(\nu + ns + \xi)^{n+4}. \quad (6)$$

We can easily find that $\phi(x/s, n) = 1 - (\nu + ns/\nu + ns + x)^{n+3}$ and so (2) gives

$$D(x/s, n, \nu) = (\nu + ns/\nu + ns + x)^{n+3} - (\nu + (n+1)s/\nu + (n+1)s + x)^{n+4}, \quad (7)$$

where instead of $D(x/s, n)$ we used $D(x/s, n, \nu)$ to indicate its dependence on ν .

Computational experience suggests that for some values of the parameters s, n, ν , $D(x/s, n, \nu)$ may be negative when $x \in [0, z]$, where z is a suitable number. For example: $D(x/7.0, 4, 1.0)$ is negative for $x < 60.56$ (and positive for $x \geq 60.5$). The validity or invalidity of Lemma 8 affects the values of the one period critical levels $\bar{x}_1(s, n+1)$ (see [1, p. 437]). In this same case, for linear ordering, holding and penalty costs with $c = 10$, $h = 5$, and $p = 30$, we get, $\bar{x}_1(7.0, 4) = 3.73 < \bar{x}_1(7.0, 5) = 4.02$. (This is not in general an expected result because $D(x/7.0, 4, 1.0)$ is not negative over $[0, \infty)$).

We note that Karlin, using (Q, n) as a sufficient statistic for the parameter ω , showed $\phi(\xi/Q, n+1) \subset \phi(x/Q, n)$ (see [3, p. 256]). Apparently the reason for the error in the lemma was the use of s instead of Q .

The procedure for obtaining the numerical results is described in the Appendix.

2.2.

In the nonadaptive inventory control problem, and with the same conditions as in Iglehart's paper, one can prove that the sequence of critical levels $\bar{x}_N$ is increasing in N . Iglehart probably based the same result for the adaptive case, on this fact, stated in Theorem 2.a, i.e., $\bar{x}_N(s, n)$ is increasing in N for any given (s, n) . To the best of our knowledge, no proof of this result has been given.

The result is apparently false. Using the density function for the demand given by (6), and linear costs with coefficients $c = 0$, $h = 2$, $p = 1000$, we computed the first two terms of the sequence $\bar{x}_N(s, n)$ (i.e. $N = 1, 2$). The result so obtained is $\bar{x}_1(25, 4) = 286.1 > \bar{x}_2(25, 4) = 276.5$ which disproves the theorem.

3. Theorems 3.e, 6.b and Lemma 2

The convergence of the sequence $\bar{x}_N(s, n)$ as $N \rightarrow \infty$ cannot be proved based on Theorem 2.a as we have seen in §2.2. Let us suppose that this has been established by some other procedure. Theorem 3.e says that $\lim_{N \rightarrow \infty} \bar{x}_N(s, n) = \bar{x}(s, n)$, where $\bar{x}_N(s, n)$ is the smallest minimizing point of the convex function $G_N(y/s, n)$ and $\bar{x}(s, n)$ is the smallest minimizing point of its limit function $G(y/s, n)$, which is also convex (see [1, p. 435]).

One can easily construct a sequence of convex functions (even strictly convex) that converges uniformly to some function, with the sequence of their smallest minimizing points converging to a limit different from the smallest minimizing point of the limit function. Since the proof of Theorem 3.e assumes the contrary, there must be some

doubt about its validity, even if we take Theorem 2.a to be true. However we note that when $\bar{x}_N(s, n)$ converges, its limit is a minimizing point for $G(y/s, n)$.

The same comments apply to Theorem 6.b.

Lemma 2 says that the sequence of functions

$$p(\xi/v, n) = q(\xi) \frac{\int_{\max(\xi, v)}^{\infty} \gamma^{n+1}(\theta) f(\theta) d\theta}{\int_v^{\infty} \gamma^n(\theta) f(\theta) d\theta}, \quad \xi \geq 0,$$

converges to the function

$$\begin{aligned} p(\xi/\hat{v}) &= \gamma(\hat{v})q(\xi) & \text{if } 0 \leq \xi \leq \hat{v}, \\ &= 0 & \text{if } \xi > \hat{v}, \end{aligned}$$

uniformly on any finite ξ interval. We observe that the limit function $p(\xi/v)$ is discontinuous as the point $\xi = \hat{v}$, so that uniform convergence is impossible.

4. Conclusions

Since Lemma 8 is true with (Q, n) instead of (s, n) as a sufficient statistic, Theorem 5.a,b,c,d, which is a consequence of this lemma must be adjusted replacing (s, n) by (Q, n) .

Since Theorem 3.e seems not to be valid, the results of Theorem 4.d and 5.d cannot be deduced as the author suggested. Despite this difficulty, these results remain valid and their proof follows through different arguments.

If we add additional assumptions to ensure the strict convexity of $L(y/s, n)$ (see [1, p. 435]) Theorem 3.e becomes true, and its proof can be deduced without requiring any monotonicity property for the sequence $\bar{x}_N(s, n)$ (see [2]). The same comment applies to Theorem 6.b.

We note that Scarf [6], when dealing with the same problem, stated the results of Theorems 3.e and 6.b but, having assumed linear ordering, holding, and penalty costs, he ensured strict convexity of $L(y/s, n)$ and so uniqueness of $\bar{x}(s, n)$ and $\bar{x}(\hat{s})$.

The invalidity of Lemma 2 does not affect any subsequent results depending on it, but some extra elaboration is needed to get their proofs.

Appendix

We can easily find that

$$G'_1(y/s, n) = c + h - (h + p)(v + ns/v + ns + y)^{n+3} \quad (8)$$

and from this

$$\bar{x}_1(s, n) = (v + ns)(A^{1/n+3} - 1) = \bar{x}_1(Q, n), \quad (9)$$

where the parameter $A = (h + p)/(h + c)$ and is greater than one in view of the condition $G'_1(0/s, n) < 0$. So

$$\begin{aligned} C'_1(x/s, n) &= -c & \text{if } x \leq \bar{x}_1(s, n), \\ &= -c + G'_1(x/s, n) & \text{if } x \geq \bar{x}_1(s, n). \end{aligned} \quad (10)$$

In view of (9), (10) we have

$$\begin{aligned} C'_1(y - \xi/(ns + \xi/n + 1), n + 1) &= -c & \text{if } \xi \geq \delta(y), \\ &= -c + G'_1(y - \xi/(ns + \xi/n + 1), n + 1) & \text{if } \xi \leq \delta(y), \end{aligned} \quad (11)$$

where $\delta(y) = (y - (v + ns)(A^{1/n+4} - 1))/A^{1/n+4}$ and $\xi \geq \delta(y)$ is the solution of the inequality $y - \xi \leq \bar{x}_1(ns + \xi/n + 1, n + 1)$. So

$$\begin{aligned} G'_2(y/s, n) &= G'_1(y/s, n) - \alpha c + \alpha \int_0^{\delta(y)} C'_1(y - \xi/(ns + \xi/n + 1), n + 1) \phi(\xi/s, n) \\ &\quad \text{if } \delta(y) > 0, \\ &= G'_1(y/s, n) - \alpha c \quad \text{if } \delta(y) \leq 0, \end{aligned}$$

where G'_1 is given by (8).

An analytic solution of equation $G'_2(y/s, n) = 0$ seems difficult to obtain. So to find $\bar{x}_2(s, n)$ we solved this equation numerically by Newton's method. The discount factor α was set equal to 1.

Note that, since backlog is permitted, x and subsequently y can take negative values. However, because we impose the condition $G'_1(0/s, n) < 0$, the critical levels $\bar{x}_N(s, n)$ cannot be negative.

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MANAGEMENT SCIENCE
Vol. 23, No. 11, July 1977
Printed in U.S.A.

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IV

A COMPUTATIONAL REFINEMENT FOR DISCRETE-VALUED DYNAMIC PROGRAMS WITH CONVEX FUNCTIONS†

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A search procedure is presented for discrete-valued dynamic programming problems with the objective of minimizing the sum of convex functions. This procedure may be more efficient than available methods if the sum is to be minimized over a uniform grid of values of the state variable. Furthermore, the procedure can be used to reduce the computational effort if only some of the return functions have a special structure.

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† Accepted by Edward J. Ignall, former Departmental Editor for Dynamic Programming and Inventory Theory; received November 1975. This paper has been with the author 2 months, for 1 revision.

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