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CONSTRUCTION OF SUBOPTIMAL CONTROL SEQUENCES*

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Summary. As an alternative to a direct solution of the Hamilton-Jacobi equation, results are presented for the determination and improvement of suboptimal controls and for obtaining bounds on the optimal value of the performance index. Treatment is restricted to the class of problems included in the well-known work of Kalman [1].

1. Introduction. It is well established that direct attempts to solve the Hamilton-Jacobi equation in order to obtain optimal feedback controls are hopeless in all but a few special cases. Bellman foresaw these difficulties in his early work on dynamic programming [2], and in addition to his main constructive method of solution, laid strong emphasis on the use of *successive approximations* for the study of optimal processes. A similar idea for obtaining suboptimal controls based on the relationship between Hamiltonian functions and performance index derivatives has been exploited by Rekasius [3] and Haussler [4]. Their work has shown promise for the special case of "stationary" problems, with a separable scalar control. The same line of reasoning will be applied here to a broader class of problems.

Consider a dynamical system represented by

$$\dot{x} = f(x, k, t), \qquad x(t_0) = x_0, \tag{1}$$

where the n-vector x is the plant state, f is a continuously differentiable n-vector function, and $k(x, t)$ is an r-vector function defined on $R^n \times R^1$. The solution of (1) will be denoted as $\phi_k(t) \triangleq \phi_k(t; x_0, t_0)$.

Let G be a closed subset of $R^n \times R^1$ to which all motions of (1) are restricted, and let the target set S be a closed subset of G. For our purposes, the function k will be called an *admissible feedback control law* if:

(a) it is continuously differentiable with values $k(x, t)$ belonging to a locally compact set $U \subset R^r$ for all t;

(b) it has the property that when substituted into (1), any motion beginning in $G - S$ reaches S, or approaches S, in a uniform asymptotic manner without leaving G.

The class of functions satisfying the above properties will be denoted as $\mathcal{K}^0$.

The terminal time $t_1 = t_1(x_0, t_0)$ will be defined as the first instant after t_0 when the motion $(\phi_k(t), t)$ becomes a member of S; or, in the asymptotic case, $t_1 = \infty$.

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The system performance is evaluated by the functional

$$J(x_0, t_0; k) = \lambda[\phi_k(t_1; x_0, t_0), t_1] + \int_{t_0}^{t_1} L[\phi_k(\alpha; x_0, t_0), k(\phi_k(\alpha; x_0, t_0), \alpha), \alpha]\, d\alpha, \tag{2}$$

where L and λ are continuously differentiable functions. We define

$$V^0(x_0, t_0) \triangleq \inf_{k \in K^0} J(x_0, t_0; k).$$

Let H be defined as

$$H(x, p, t, u) \triangleq \langle f(x, u, t), p \rangle + L(x, u, t), \tag{3}$$

where p is an n-vector, u is an r-vector and $\langle\, , \rangle$ denotes the inner product. Assume that H has a unique absolute minimum for each x, p, and t with respect to the values $u \in U$, and let the associated location of the minimum be denoted as $c(x, p, t)$. Assuming that c is a continuously differentiable function of x, p and t, we define the Hamiltonian as

$$H^0(x, p, t) \triangleq H(x, p, t, c(x, p, t)) = \min_{u \in U} H(x, p, t, u) \tag{4}$$

and the Hamilton-Jacobi equation as

$$V_t + H^0(x, V_x, t) = 0, \tag{5}$$

where $V(x, t)$ is a scalar function defined on $R^n \times R^1$, $V_t = \partial V/\partial t$ and $V_x = \operatorname{grad} V$. Kalman [1] has shown that if $V(x, t)$ is twice continuously differentiable in all arguments, if it satisfies (5) in G and the boundary condition $V(x, t) = \lambda(x, t)$ on S, and in addition if the function $k^0(x, t) = c(x, V_x(x, t), t)$ is admissible, then $V(x, t) = V^0(x, t)$.

The solution of (1) is easily seen to have the property that

$$\phi_k(\alpha; \phi_k(t; x_0, t_0), t) = \phi_k(\alpha; x_0, t_0) = \phi_k(\alpha); \tag{6}$$

and furthermore, if $(x_0, t_0) \in G - S$, we have for the terminal time

$$t_1(\phi_k(t; x_0, t_0), t) = t_1(x_0, t_0), \qquad t_0 \leqq t \leqq t_1(x_0, t_0). \tag{7}$$

It follows from (2), (6) and (7) that

$$J(\phi_k(t; x_0, t_0), t; k) = \lambda[\phi_k(t_1; x_0, t_0), t_1] + \int_{t}^{t_1} L[\phi_k(\alpha; x_0, t_0), k(\phi_k(\alpha; x_0, t_0), \alpha), \alpha]\, d\alpha \tag{8}$$

for $t_0 \leqq t \leqq t_1$. Consequently, the Eulerian derivative of J along motions of (1) is given by

$$\frac{dJ}{dt}(\phi_k(t; x_0, t_0), t; k) = -L[\phi_k(t; x_0, t_0), k(\phi_k(t; x_0, t_0), t), t] \tag{9}$$

or, denoting $\phi_k(t; x_0, t_0)$ by x, we have in more casual notation

$$\dot{J}(x, t; k) = -L(x, k, t). \tag{10}$$

Whenever there is no chance of confusion we will use the latter representation of the result, it being understood that (9) is implied.

2. Transformations of the successive approximation. Given any optimal feedback control problem described above, define $\mathcal{V}$ as the set of all continuously differentiable functions $V: R^n \times R^1 \to R^1$ such that $V(x, t) = \lambda(x, t)$ on S. Let $\mathcal{V}^0$ be the subset of $\mathcal{V}$ such that if $k(x, t) = c(x, V_x(x, t), t)$ then $k \in \mathcal{K}^0$, i.e., k is admissible. Note that by our assumptions, if it exists, $V^0 \in \mathcal{V}^0$.

Next we define the basic transformations used to generate suboptimal control sequences.

(a) $T_1 : \mathcal{V}^0 \to \mathcal{K}^0$ is defined for any $V \in \mathcal{V}^0$ by $T_1(V) = k$, where $k(x, t) = c(x, V_x, t)$.

(b) $T_2 : \mathcal{K}^0 \to \mathcal{V}$ is defined by $T_2(k) = V$, where $V(x, t) = J(x, t; k)$. Clearly $J(x, t; k) = \lambda(x, t)$ on S, and since λ, L, and k are continuously differentiable, so is V.

(c) $T: \mathcal{V}^0 \to \mathcal{V}$ is the composite mapping defined for $V \in \mathcal{V}^0$ by $T(V) = T_2(T_1(V)) = J(x, t; k)$ with $k(x, t) = c(x, V_x, t)$.

3. Development of the basic inequalities.

LEMMA 1. *Suppose $V \in \mathcal{V}^0$ and $W \in \mathcal{V}$. Then, if*

$$H^0(x, V_x, t) + V_t \leqq H^0(x, W_x, t) + W_t, \qquad (x, t) \in G - S, \tag{11}$$

it follows that

$$W(x, t) \leqq V(x, t), \qquad (x, t) \in G. \tag{12}$$

Proof. Since $V \in \mathcal{V}^0$ the control $k(x, t) = c(x, V_x, t)$ causes the motion of (1) to enter S from any initial phase in G. Taking derivatives along this motion we have

$$\begin{aligned}\dot{V} - \dot{W} &= \langle f(x, k, t), V_x\rangle - \langle f(x, k, t), W_x\rangle + V_t - W_t \\ &= [H^0(x, V_x, t) + V_t - H^0(x, W_x, t) - W_t] \\ &\qquad + [H^0(x, W_x, t) - H(x, W_x, t, k)] \\ &\leqq 0,\end{aligned}$$

since the term in the first bracket is nonpositive by assumption, whereas the term in the second bracket is nonpositive by the definition of H^0 in (4). It follows that along the motion of the indicated system $\dot{V}(x, t) \leqq \dot{W}(x, t)$. Noting that both Eulerian derivatives are taken along the same motion and therefore with a common termination point on S, a simple integration yields (12).

THEOREM 1. *Let* $V \in \mathcal{V}^0$ *and* $W \in \mathcal{V}$. *Then for* $(x, t) \in G$,

$$H^0(x, V_x, t) + V_t \leqq 0 \text{ implies } V^0(x, t) \leqq V(x, t), \tag{13}$$
$$H^0(x, W_x, t) + W_t \geqq 0 \text{ implies } W(x, t) \leqq V^0(x, t).$$

Proof. The conclusion follows from Lemma 1 together with the fact that $V^0 \in \mathcal{V}^0$ and $H^0(x, V_x^0, t) + V_t^0 = 0$. Note that all of the above results also hold for strict inequalities. Theorem 1 provides a basic means of determining upper and lower bounds on $V^0(x, t)$.

The next result gives an alternate method of evaluating the performance index for a given admissible control law.

LEMMA 2. *Let* $W \in \mathcal{V}$ *and* $k \in \mathcal{K}^0$. *Then*

$$W = T_1(k) \text{ if and only if } H(x, W_x, t, k) + W_t = 0 \quad \text{in } G - S. \tag{14}$$

Proof. Assume first that $H(x, W_x, t, k) + W_t = 0$. Taking the derivative of $W(x, k)$ along motions of (1) with the control k we have

$$\dot{W} = H(x, W_x, t, k) + W_t - L(x, k, t) = -L(x, k, t);$$

but from (10), $\dot{J} = -L(x, k, t)$. Noting that the restrictions on f, k and L imply J is continuously differentiable in G and integrating along the motion of (1) yield $W(x, t) = J(x, t; k)$, i.e., $W = T_1(k)$. Conversely, $W = T_1(k)$ implies $\dot{W} = -L$ which leads to $H(x, W_x, t, k) + W_t = 0$.

Let us now turn to the composite mapping $T = T_2T_1$.

THEOREM 2. *Let* $V \in \mathcal{V}^0$ *and* $W \in \mathcal{V}$. *Then*

$$W = T(V) \text{ if and only if } H(x, W_x, t, c(x, V_x, t)) + W_t = 0 \quad \text{in } G - S. \tag{15}$$

The proof follows directly from Lemma 2 and the smoothness assumption on $c(x, V_x(x, t), t) = k(x, t) = T_1(V)(x, t)$.

Notice that Lemma 2 and Theorem 2 both give methods of determining $W \in \mathcal{V}$ such that $H(x, W_x, t, k) + W_t = 0$ and thus $H^0(x, W_x, t) + W_t \leqq 0$.

If it turns out that $W \in \mathcal{V}^0$, the function takes on a usefulness which is now indicated.

THEOREM 3. *Let* $W \in \mathcal{V}^0$ *and* $W^* = T(W)$. *Then*

$$H^0(x, W_x, t) + W_t \leqq 0 \quad \text{in } G - S \tag{16}$$

implies that

$$V^0(x, t) \leqq W^*(x, t) \leqq W(x, t). \tag{17}$$

Proof. Taking derivatives of W and W^* along the same motion of (1) with $k(x, t) = c(x, W_x, t)$ we have

$$\dot{W} = \langle f(x, k, t), W_x \rangle + W_t = H^0(x, W_x, t) + W_t - L(x, k, t) \leqq -L(x, k, t),$$

whereas from (10), $\dot{W}^* = \dot{J}(x, t; k) = -L(x, k, t)$. The desired result follows by integration.

It is easy to see that V^0 is a fixed point of the mapping T, but the converse is not obvious.

THEOREM 4. *If V^0 exists, then V^0 is a fixed point of T, i.e., $V^0 = T(V^0)$. Conversely, if V is a fixed point, then $V(x, t) = V^0(x, t)$ on G.*

Proof. If V^0 exists, then $V^0 \in \mathcal{V}^0$. Let $V^* = T(V^0)$. From Theorem 3 we have $V^0(x, t) \leqq V^*(x, t) \leqq V^0(x, t)$, so $V^* = V^0$. Conversely, assume $V = T(V)$; then by Theorem 2, $H(x, V_x, t, k) + V_t = 0$, where $k(x, t) = c(x, V_x, t)$, i.e., $H^0(x, V_x, t) + V_t = 0$ or V is a solution of the Hamilton-Jacobi equation. With the assumption that V is a member of $\mathcal{V}^0$, the domain of T, we have $V = V^0$.

Note that if V^0 exists, it is unique.

COROLLARY 1. *If V^0 exists, there is one and only one fixed point of T in $\mathcal{V}^0$.*

4. Iterations and convergence conditions.

The question now arises as to whether the process outlined in Theorem 3 can be continued. Given any $V^1 \in \mathcal{V}^0$ we establish the successive approximation procedure through T, i.e., $V^{n+1} = T(V^n)$, by making the following assumption.

ASSUMPTION. $T(V^n) \in \mathcal{V}^0$ for $n = 1, 2, 3, \cdots$.

THEOREM 5. *If $V^1 \in \mathcal{V}^0$ and $V^{n+1} = T(V^n)$, $n = 1, 2, 3, \cdots$, then*

$$V^0(x, t) \leqq V^{n+1}(x, t) \leqq V^n(x, t) \leqq V^1(x, t), \qquad (x, t) \in G.$$

Proof. Let $k^{n+1}(x, t) = c(x, V_x^n, t) = T_1(V^n)$, and note that we have defined $V^n(x, t) = J(x, t; k^n) = T_2(k^n)$. By Theorem 2 we have for each n

$$H(x, V_x^n, t, k^n) + V_t^n = 0;$$

thus $H^0(x, V_x^n, t) + V_t^n \leqq 0$, and the conclusion follows inductively from Theorem 3.

Theorem 5 indicates that the estimate of $V^0(x, t)$ can only be improved by the successive approximations. We may state stronger results if further restrictions are imposed.

THEOREM 6. *If the iteration procedure terminates in a finite number of steps, that is, if $T(V^n) = V^n$ for some finite n, then V^0 exists and $V^0(x, t) = V^n(x, t)$ on G.*

Proof. Since V^n is a fixed point in $\mathcal{V}^0$, the result follows directly from Theorem 4.

If the iteration does not end in a finite number of steps, Lemma 3 follows from Theorem 5.

LEMMA 3. *For every sequence* $\{V^n\}$, *there exists a function* V^* *such that* $V^n(x, t) \downarrow V^*(x, t)$ *pointwise on* G. *If* G *is bounded, the convergence is uniform.*

In order that $V^* = V^0$, it is required that $V^* \in \mathcal{V}^0$. If in addition T is continuous at V^*, the condition $V^* = V^0$ is assured. In order to make the statement more precise, let us assume G is bounded and define a distance function on $\mathcal{V}$ as

$$d(V_1, V_2) \triangleq \sup_{(x,t) \in G} \{| V_1(x, t) - V_2(x, t) |\} \quad \text{for } V_1, V_2 \in \mathcal{V}.$$

THEOREM 7. *Let* T *be continuous in* $\mathcal{V}^0 \subset \mathcal{V}$. *If* $\{V^n\}$ *is such that* $T(V^n) = V^{n+1}$ *and* $V^n(x, t) \downarrow V^*(x, t)$ *as in the above mentioned construction, and if* $V^* \in \mathcal{V}^0$, *then* $V^* = V^0$.

Proof. With the defined metric, Lemma 3 implies that $V^n \to V^*$ and $T(V^n) \to V^*$. Since T is continuous, $T(V^*) = V^*$, so that by Theorem 4, $V^* = V^0$.

Above, we give conditions under which the iteration will converge to the optimal $V^0(x, t)$. Perhaps more important from a practical standpoint is that the successive approximations are monotone decreasing under more general conditions and a gauge can be obtained by finding *lower* bounds on $V^0(x, t)$ by the use of Theorem 1.

5. Quasilinearization and the canonical equations. It is interesting to note that, by Theorem 2, evaluating the successive approximation $V^{n+1} = T(V^n)$ amounts to solving a sequence of *linear* partial differential equations of the first order, since

$$H(x, V_x^{n+1}, t, c(x, V_x^n, t)) + V_t^{n+1} = 0$$

implies

$$\langle V_x^{n+1}, f(x, c(x, V_x^n, t), t)\rangle + V_t^{n+1} = -L(x, c(x, V_x^n, t), t). \tag{18}$$

It is well known [1], [5] that the Hamilton-Jacobi equation $H^0(x, V_x, t) + V_t = 0$ specifies a two point boundary value problem in the canonical equations $\dot{x} = H_p^0$, $\dot{p} = -H_x^0$. Because of their general nonlinearity, numerical integration of the canonical equations presents a formidable problem. Bellman and Kalaba [6] have employed quasilinearization to reduce problems of this type to problems of solving sequences of *linear* two-point boundary value problems.

As one might expect, the characteristic equations associated with (18) provide a similar formal mechanism of solution. They may be written with $\phi^n(t) \triangleq \phi_{k^n}(t)$ as

$$\text{(19)} \qquad \begin{aligned} \dot\phi^{n+1}(t) &= H_p(\phi^{n+1}, p^{n+1}, t, c(\phi^n, p^n, t)), \\ -\dot p^{n+1}(t) &= H_x(\phi^{n+1}, p^{n+1}, t, c(\phi^n, p^n, t)), \end{aligned}$$

or, more explicitly,

$$\text{(20)} \qquad \begin{aligned} \dot\phi^{n+1}(t) &= f(\phi^{n+1}, c(\phi^n, p^n, t), t), \\ -\dot p^{n+1}(t) &= f_x(\phi^{n+1}, c(\phi^n, p^n, t), t)p^{n+1} + L_x(\phi^{n+1}, c(\phi^n, p^n, t), t) \\ &\quad + c_x(\phi^n, p^n, t)[L_u(\phi^{n+1}, c(\phi^n, p^n, t), t) \\ &\quad + f_u(\phi^{n+1}, c(\phi^n, p^n, t), t)p^n], \end{aligned}$$

where f_x, c_x, f_u are Jacobian matrices and L_x, L_u are gradient vectors. It is seen that the second equation is linear in p^{n+1}. The equations may be integrated iteratively by choosing an appropriate admissible control $u(t)$ in place of $c(\phi^1(t), p^1(t), t)$. This amounts to an "approximation in policy space" [2]. Boundary conditions are obtained from the general transversality condition

$$\text{(21)} \qquad [d\lambda + H^0\, dt + \langle p, dx\rangle]_{t_0}^{t_1} = 0,$$

where the differentials are consistent with the side constraints. We do not wish to go further into these matters, but mention them to show connections with other work in the field.

6. Applications.

Example 1. A *norm invariant* system may be given by

$$\dot x = A(x)x + k, \quad \text{where} \quad A + A^T = 0.$$

Let $S = \{(x, t) \mid \|x\| = \theta\}$, $G = \{(x, t) \mid \|x\| \geqq \theta\}$, $U = \{k \mid \|k\| \leqq 1\}$, $\lambda = 0$, and $L = 1$, where θ is any (small) positive number. Since without control all solutions are on the constant norm surface $\|x\| = \|x_0\|$, it is reasonable to assume that $V^1(x, t) = g(\|x\|)$, where $dg(\alpha)/d\alpha > 0$. It is easy to show that $c(x, p, t) = -p/\|p\|$; consequently, $k^2(x, t) = c(x, V_x^1, t) = -x/\|x\|$. It is then straightforward to show that $V^3(x, t) = V^2(x, t) = \|x\| - \theta$. Therefore, by Theorem 6, $V^0(x, t) = \|x\| - \theta$, and the optimal control law is $k^0(x, t) = -x/\|x\|$.

Example 2. In the *first-order linear* system

$$\dot x = k,$$

with $L = x^2 + k^2$, $G = U = R^1$, $S = \{(x, t) \mid x = 0\}$, the well-known solution is $V^0(x, t) = x^2$, $k^0(x, t) = -x$. If we assume $V^n(x, t) = Kx^2$, then

(18) becomes

$$-KxV_x^{n+1} + x^2 + K^2x^2 = 0;$$

so $V^{n+1}(x, t) = (1 + K^2)x^2/(2K)$, and it is seen that the related sequence converges to $V^0(x, t)$.

Example 3. A simple, but explicitly insolvable minimum time problem is defined by letting $L = 1$, $G = \{(x, t) \mid \| x \| \geqq \theta\}$, $S = \{(x, t) \mid \| x \| = \theta\}$, and $U = \{k \mid \| k \| \leqq 1\}$ for the system

$$\dot{x}_1 = -x_1 + k_1 ,$$

$$\dot{x}_2 = -2x_2 + k_2 .$$

The small parameter θ should be held to a positive value in order for the theory to apply, but it seems worthwhile to proceed formally with $\theta = 0$ in the interest of simpler expressions. Application of Theorem 1 shows that

$$\log (1 + 2 \| x \|)^{1/2} \leqq V^0(x, t) \leqq \log (1 + \| x \|).$$

Beginning the iteration of Theorem 5 with $V^1(x, t) = \log (1 + \| x \|)$ we obtain $k^2(x, t) = -x/\| x \|$. Evaluation of $V^2 = T(V^1)$ by means of (18) shows that

$$V^2(x, t) = \log (1 + 2 \| x \| + x_1^2)^{1/2},$$

$$k^3(x, t) = \frac{-(x_1(1 + \| x \|), x_2)}{[\| x \|^2 + x_1^2 \| x \|^2 + 2 \| x \| x_1^2]^{1/2}} .$$

From the latter control law, $V^3(x, t)$ may be evaluated numerically as the time it takes to reach the origin.

Letting $W(x, t) = (1 + 2 \| x \|)^{1/2}$, we have

$$W(x, t) \leqq V^0(x, t) \leqq V^3(x, t) \leqq V^2(x, t) \leqq V^1(x, t).$$

The $t = 1$ isochrones for these quantities are shown in Fig. 1. As mentioned above, the set of points where $V^3(x, t) = 1$ was determined by numerical integration. Actually, the set of points satisfying $V^0(x, t) = 1$ in the first quadrant can be specified parametrically by $x_1 = (1/\beta) \log [\beta e + \sqrt{1 + \beta^2 e^2}]$, $x_2 = [e\sqrt{1 + \beta^2 e^2} - \sqrt{1 + \beta^2} - x_1]/(2\beta)$, where β is a parameter ranging over the real line. A comparison shows that the approximation $V^3(x, t) = 1$ is accurate to within a few tenths of a percent.

Example 4. Here we consider a class of first-order problems described by $\dot{x} = g(x) + k$, with $L(x, k, t) = \alpha(x) + k^2$, $G = \{(x, t) \mid x \geqq 0\}$, $U = R^1$, $S = \{(x, t) \mid x = 0\}$. We assume that $\alpha(x)$ is positive definite.

In this instance it is possible to carry out the complete approximation process in the policy space $\mathcal{K}^0$ by using the transformation $T^* = T_1T_2$

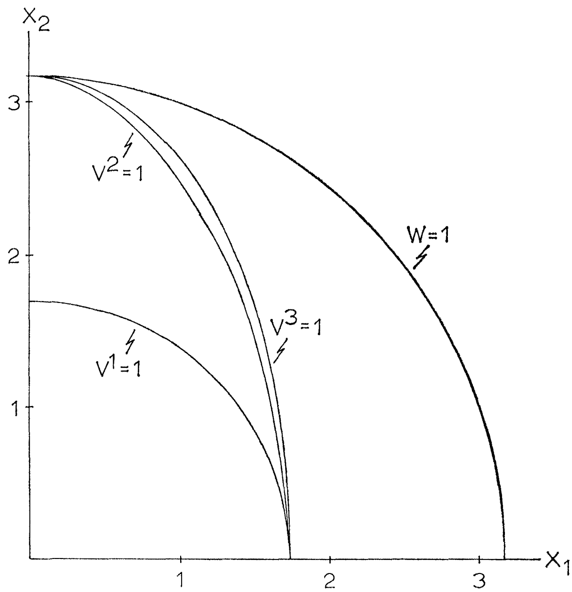

FIG. 1. *Isochrones corresponding to $t = 1$ for the successive approximations of Example* 3

rather than $T = T_2T_1$ as has been discussed in the text. We have $c(x, V_x, t) = -\frac{1}{2}V_x$, and (18) becomes

(22) $$V_x^{n+1}(g - \tfrac{1}{2}V_x^n) + \tfrac{1}{4}(V_x^n)^2 + \alpha = 0.$$

Further, if $V^n \in \mathcal{V}^0$, we have by Theorem 1

(23) $$V_x^n g - \tfrac{1}{4}(V_x^n)^2 + \alpha \leqq 0, \qquad x \geqq 0.$$

Substituting $k^{n+1} = -\frac{1}{2}V_x^n$ into (22) yields

$$k^{n+1} = T^*(k^n) = \frac{1}{2}\frac{(k^n)^2 + \alpha}{k^n + g},$$

or

$$k^{n+1} + g = \frac{1}{2}\left[(k^n + g) + \frac{\alpha + g^2}{(k^n + g)}\right].$$

Now if k^n is admissible, $k^n + g < 0$ for $x > 0$ and k^{n+1} is continuously differentiable for $x > 0$. Furthermore, the origin is asymptotically stable for $\dot{x} = g(x) + k^{n+1}$, since from (23)

$$\dot{V}^n = gV_x^n + k^{n+1}V_x^n = gV_x^n - \tfrac{1}{2}(V_x^n)^2 \leqq -\alpha - \tfrac{1}{4}(V_x^n)^2,$$

and thus V^n is a Liapunov function for the system. Therefore, k^{n+1} is admissible. It is easy to show that

$$k^{n+1} + g \to -\sqrt{\alpha + g^2}, \quad \text{or} \quad k^n \to -g - \sqrt{\alpha + g^2} = k^0.$$

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