[4] J. L. Doob, "Topics in the Theory of Markoff Chains," *Trans. Am. Math. Soc.*, **52**, 31–64, 1942.

[5] D. Ray, "Stationary Markov Processes with Continuous Paths," *Trans. Am. Math. Soc.* ,**82**, 452–493, 1956.

[6] Added in proof. It appears that Chung's result that $y(t, w) \in \mathfrak{M}$ on $t \geqq 0$ with respect to the triple $(P(\cdot 1\Delta), \Delta F, \Delta)$ where $\Delta = \cup_j \Delta_j = \cup_j [w; x(\alpha(w), w) = j]$ will follow using our proof if one chooses the approximating $\alpha_n(w)$ more delicately. Let $R_p = [2^{-\frac{m}{p}}; m = 1, 2 \ldots]$ for $p = 1, 2 \ldots$ and let $R = \cup_p R_p$ be a separability set for $x(t, w)$. Since $x(t, w) \in \mathfrak{M}^*$ the set $[w; w \in \Delta_j, \alpha(w) < t] \in \mathfrak{F}_t\{x(s), s \leq t\}$. Let $\pi_p[0 = a_{1,p} < a_{2,p} < \ldots]$ be a sequence of partitions of $[0, \infty)$ with norms tending to 0 and define the optional random variables as $\alpha_p^*(w) = a_k$ on $B_k = [w; a_{k-1} < \alpha \leq a_k]$ with left inclusion at $k = 1$. Fix integers j, k and p and let $A_i = A_i(k, j) = [w; w \in \Delta_j \cap B_k, \alpha(w) < s_i, x(s_i, w) = j]$ where $s_1 < s_2 \ldots < s_q$ is an ordering of $R_p \cap (a_{k-1,p}, a_{k,p}]$. We define $\alpha_p(w) = s_l$ on $A_l - \bigcup_{l=1}^{l-1} A_1$, $l = 1, 2 \ldots q$; the r.v.'s are defined similarly for each k and j and we let $\alpha_p(w) = \alpha_p^*(w)$ elsewhere. The r.v.'s $\alpha_p(w)$ are optional and satisfy $\alpha_p(w) \downarrow \alpha(w)$ and $x(\alpha_p, w) \rightarrow x(\alpha, w)$ on Δ; hence we may let $t_1 = t^* = 0$ on the triple $(P(\cdot 1\Delta), \Delta F, \Delta)$ in the proof of our theorem and obtain the stronger result.

Finally we mention that Jushkevich has quite recently (*Russian J. of Prob.*) obtained results on the strong Markov property in a different form from that presented here. Also Chung has employed the continuity of the conditional distribution of $y(t)$ relative to α to show that $y(t)$ is separable without modification.

DYNAMIC PROGRAMMING, SUCCESSIVE APPROXIMATIONS, AND MONOTONE CONVERGENCE

BY RICHARD BELLMAN

RAND CORPORATION, SANTA MONICA, CALIFORNIA

Communicated by Philip M. Morse, April 21, 1958

1. *Introduction.*—Our object in this paper is to show that a blend of dynamic programming, successive approximations, and digital computers enables us to approach various classes of variational problems formerly far beyond our reach.

To illustrate the application of these methods, we shall consider two problems. The first is that of minimizing the functional

$$J(v) = \int_0^T F(x_1, x_2, \ldots, x_N)dt + G(x_1(T), x_2(T), \ldots, x_N(T)) \qquad (1.1)$$

over all forcing functions $v_i(t)$ related to the $x_i(t)$ by means of relations of the type

$$\frac{dx_i}{dt} = H_i(x_1, x_2, \ldots, x_N) + v_i(t), \qquad x_i(0) = c_i, \qquad i = 1, 2, \ldots, N, \qquad (1.2)$$

and subject to the constraints of the form

$$\begin{aligned} &(a)\ \int_0^T K_j(v_1, v_2, \ldots, v_N)dt \leq b_j, \qquad j = 1, 2, \ldots, L, \\ &(b)\ p_i(t) \leq v_i(t) \leq q_i(t), \qquad 0 \leq t \leq T, i = 1, 2, \ldots, N. \end{aligned} \qquad (1.3)$$

The second problem is a generalized Hitchcock-Koopmans transportation problem. It involves the minimization of the function $C(x) = \sum g_{ij}(x_{ij})$ subject to the constraints

$$\sum_{i=1}^{N} x_{ij} = s_i, \qquad \sum_{i=1}^{M} x_{ij} = r_j, \qquad x_{ij} \geq 0. \tag{1.4}$$

As we have shown elsewhere,[1] the computational solution of questions of this type can be transformed by means of the functional equation technique of the theory of dynamic programming into problems involving the recursive determination of sequences of functions of N variables.

If $N \geq 3$, the limited memory and speed of current digital computers prevents a routine application of these techniques at the present time. Hence, if we wish to attack general problems of this nature, we must introduce more refined and subtle methods.

2. *Successive Approximation and Monotonicity.*—The starting point of our investigation is the result contained in an earlier paper,[2] where it was shown that if the equations of (1.2) are linear, and if the functions F and G are linear, then the variational problem described in the preceding section can be resolved in terms of sequences of functions of one variable, regardless of the value of N.

Furthermore, it was shown if the equations of (1.2) are linear, H is linear, but G is a non-linear function of only k of the components of $x(T)$, then a computational solution can be effected in terms of sequences of functions of k variables.

These facts lead us to contemplate the use of successive approximations; an extensive discussion will be found in Bellman.[3] Here we wish to present a simple argument which establishes the monotone character of the approximation scheme we shall discuss below. There are a number of quite interesting questions concerning convergence to the absolute minimum which we shall treat at another time.

As we have shown,[3] we can consider, by means of a suitable transformation, that F is zero. Let us consider here only the case where G is linear in the $x_i(T)$. Under these assumptions, let $v_i^{(0)}$ be a first approximation to the solution of the variational problem, and let $x_i^{(0)}$ be the state functions determined from (1.2) using these values of the $v_i^{(0)}$. To determine a second approximation, consider the problem of minimizing $G(x(T))$ over all functions satisfying the constraints of (1.3), where the x_i are related to the v_i by means of the linear equations

$$\begin{aligned} \frac{dx_i}{dt} = H_i(x_1^{(0)}, \ldots, x_N^{(0)}) &+ \sum_{j=1}^{N} (x_j - x_j^{(0)})\partial H_i/\partial x_j^{(0)} \\ &+ v_i(t), \qquad x_i(0) = c_i, \qquad i = 1, \ldots, N. \end{aligned} \tag{2.1}$$

As pointed out above, this problem can be solved computationally in terms of sequences of functions of one variable. In some cases, in particular, when the constraint of (1.3a) is not present, the problem is readily solved analytically.[4,5]

Once the new forcing functions $v_i^{(1)}$ have been obtained in this way, new state functions $x_i^{(1)}$ are determined from (1.2). Repeating this process step by step, we determine a sequence of forcing functions $\{v_i^{(k)}\}$ and a sequence of state functions $\{x_i^{(k)}\}$. Let us now show that we have monotone approximation in the sense that

$$\begin{aligned} G(x_1^{(0)}(T), \ldots, x_N^{(0)}(T)) &\geq G(x_1^{(1)}(T), \ldots, x_N^{(1)}(T)) \geq \ldots \\ &\geq G(x_1^{(k)}(T), \ldots, x_N^{(k)}(T)) \geq \ldots. \end{aligned} \tag{2.2}$$

To prove the first inequality, observe that if in (2.1), $v_i(t)$ is taken equal to

$v_i^{(0)}(t)$, we obtain a system of linear equations whose solution is clearly $x_i = x_i^{(0)}$. It follows that a set of forcing functions which minimize G subject to the linear equations of (2.1), together with the original constraints, yields a value of G which is at most $G(x_1^{(0)}(T), \ldots, x_N^{(0)}(T))$. The general result follows inductively.

This monotonicity is not surprising, since we are using the technique of approximation in policy space.[1]

3. *Successive Approximation and the Hitchcock-Koopmans Problem.*—Let us now turn to the second problem described in section 1. As a first approximation, let x_{ij} be a set of values satisfying the constraints in (1.4). To obtain a second approximation, we fix the quantities sent out from the sources $i = 3$ to $i = N$, and determine the allocations from the first two sources so as to minimize the cost of supplying the remaining demand. This problem can be resolved in terms of sequences of functions of one variable.[6]

To obtain a third approximation, we fix the allocations from the first source and the sources $i = 4$ to $i = N$, and determine the allocation from the second and third sources so as to minimize the cost of supplying the remaining demands.

Continuing in this fashion, we obtain a sequence of problems, each of whose solutions depends upon a sequence of functions of one variable. As above, it is easy to see that the sequence of costs obtained in this way is monotone decreasing. Once again, interesting questions arise concerning convergence which we do not enter into here.

Similar techniques can be applied to other classes of combinatorial problems as will be shown elsewhere.

[1] R. Bellman, *Dynamic Programming* (Princeton: Princeton University Press, 1957).

[2] R. Bellman, "Terminal Control, Time Lags, and Dynamic Programming," these PROCEEDINGS, **43**, 927–930, 1957.

[3] R. Bellman, "Some New Techniques in the Dynamic Programming Solution of Variational Problems," *Quart. Appl. Math.* (to appear).

[4] R. Bellman, I. Glicksberg, and O. Gross, "On Some Variational Problems Occurring in the Theory of Dynamic Programming," *Rend. Palermo*, Ser. II, **3**, 1–35, 1954.

[5] R. Bellman, W. H. Fleming, and D. V. Widder, "Variational Problems with Constraints," *Ann. di Mat.*, Ser. IV, **49**, 301–323, 1956.

[6] R. Bellman, "Notes on the Theory of Dynamic Programming—Transportation Models," *Management Sci.*, **4**, 191–195, 1958.

---

*ON THE THEOREM OF BERTINI FOR LOCAL DOMAINS**

BY WEI-LIANG CHOW

DEPARTMENT OF MATHEMATICS, THE JOHNS HOPKINS UNIVERSITY

Communicated by N. E. Steenrod, August 26, 1957

1. The well-known Theorem of Bertini on reducible linear systems of divisors in an algebraic variety asserts that any such system, assumed to be free from fixed components, is composite with a pencil. A somewhat special form of this theorem, which is actually a special case of it, states that if a linear system of divisors is obtained from a rational transformation of the variety onto a projective