

# Neurodynamic Programming and Zero-Sum Games for Constrained Control Systems

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**Abstract**—In this paper, neural networks are used along with two-player policy iterations to solve for the feedback strategies of a continuous-time zero-sum game that appears in  $L_2$ -gain optimal control, suboptimal  $H_\infty$  control, of nonlinear systems affine in input with the control policy having saturation constraints. The result is a closed-form representation, on a prescribed compact set chosen *a priori*, of the feedback strategies and the value function that solves the associated Hamilton–Jacobi–Isaacs (HJI) equation. The closed-loop stability,  $L_2$ -gain disturbance attenuation of the neural network saturated control feedback strategy, and uniform convergence results are proven. Finally, this approach is applied to the rotational/translational actuator (RTAC) nonlinear benchmark problem under actuator saturation, offering guaranteed stability and disturbance attenuation.

**Index Terms**—Actuator saturation,  $H_\infty$  control, policy iterations, zero-sum games.

## I. INTRODUCTION

NEURODYNAMIC programming [9], also known as approximate dynamic programming, has been the subject of many studies and research. At the heart of this area is the approximation of optimal control policies and value functions for complex dynamical systems. Most of the preliminary results focused on dynamic programming problems in discrete-time involving discrete-action and discrete-state spaces. Recently, generalizations to dynamical systems that are common in control engineering have received a lot of attention [37]. In this paper, we contribute new results addressing zero-sum games that are related to the  $H_\infty$  control problem. In our recent work [2], we introduced a new Hamilton–Jacobi–Isaacs (HJI) equation that is tailored to constrained input systems. The HJI equation was formulated using performance functionals with quasi-norms to encode input constraints. As for the case of unconstrained inputs [39], once the game value function of the HJI

equation is smooth and computed, a feedback controller can be synthesized that results in closed-loop asymptotic stability and provides  $L_2$ -gain disturbance attenuation. Computing the value function of the constrained input zero-sum game by solving the HJI equation is a formidable task. In [2], we have introduced a two-player policy iteration scheme that results in a framework that allows the use of neural networks to approximate optimal policies and value functions. We have proved the convergence of a two-player policy iterations scheme to solve zero-sum games for constrained control input. The main contribution of this paper is to build on the results in [2] and to systematically show how we may synthesize closed-form solutions for the complicated dynamical programming problems considered in [2], i.e., zero-sum games with constrained inputs.

Most previous work on solving HJI equations is related to affine in the input nonlinear systems with no constraints on the input signal. For such unconstrained affine in input nonlinear systems, a direct approach to solve the HJI equation is given by the third coauthor in [18] and [19], where the assumed smooth solution is found by solving for the Taylor series expansion coefficients in a very efficient and organized manner. In [8], an indirect method to solve the HJI equation for unconstrained systems based on policy iterations is proposed where the solution of a sequence of differential equations, linear in the associated cost, converges to the solution of the related HJI equation, which is nonlinear in the available storage. Galerkin techniques are used to solve the sequence of linear differential equations, resulting in a numerically efficient algorithm that, however, requires computing numerous integrals over a well-defined region of the state space.

In [2], policy iterations were proposed to solve the constrained-input HJI equation. In this paper, we build on the results in [2] by using function approximation techniques to approximate the involved value functions and policies. This is an two-player extension to our earlier neural network policy iteration approach to solve the constrained-input HJB equations [1].

The use of neural networks in feedback control has been very successful and rigorously studied by many researchers. Several successful neural network controllers have been reported by Chen and Liu [12], Lewis *et al.* [24], Polycarpou [32], Rovithakis and Christodoulou [33], Seshagiri and Khalil [36], Sadegh [34], Ge *et al.* [16], and Sanner and Slotine [35]. It has been shown that neural networks can effectively extend adaptive control techniques to nonlinearly parameterized systems. Applications of neural networks to optimal control, adaptive critics were first proposed by Werbos in [27]. Parisini and Zoppoli [31] used neural networks to derive optimal control

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laws for discrete-time stochastic nonlinear system. The status of neural network control as of 2001 appears in Narendra and Lewis [29].

The importance of this paper stems from the fact that we are providing a practical solution method based on neural networks to solve for suboptimal  $H_\infty$  control of constrained input systems. The  $H_\infty$  norm has played an important role in the study and analysis of robust optimal control theory since its original formulation in an input–output setting by Zames [42]. Earlier solution techniques involved operator-theoretic methods. State–space solutions were rigorously derived in [21] for the linear system case that required solving several associated Riccati equations. Later, more insight into the problem was given after the  $H_\infty$  linear control problem was posed as a zero-sum two-person differential game by Başar [6]. The nonlinear counterpart of the  $H_\infty$  control theory was developed by Van der Schaft [39]. He utilized the notion of dissipativity introduced by Willems [40], [41] and formulated the  $H_\infty$  control theory into a nonlinear  $L_2$ -gain optimal control problem. The  $L_2$ -gain optimal control problem requires solving a Hamilton–Jacobi equation, namely, the HJI equation. Conditions for the existence of smooth solutions of the Hamilton–Jacobi equation were studied through invariant manifolds of Hamiltonian vector fields and the relation with the Hamiltonian matrices of the corresponding Riccati equation for the linearized problem, [39]. Later, some of these conditions were relaxed by Isidori and Astolfi [20] into critical and noncritical cases.

The remainder of this paper is organized as follows. Section II reviews the constrained input suboptimal  $H_\infty$  control problem and the policy iteration approach proposed in [2]. In Section III, the novel results of this paper appear, where a neural network least-squares-based algorithm is described to practically solve for the constrained input HJI equation. Section IV demonstrates the stability and convergence of the proposed neural network algorithm. Section V illustrates a successful application of the proposed algorithm to the rotational/translational actuator (RTAC) nonlinear benchmark problem under actuator saturation originally proposed in [11]. Comments and conclusions are given in Section VI.

## II. $L_2$ -GAIN OPTIMAL CONTROL OF CONSTRAINED INPUT SYSTEMS WITH POLICY ITERATIONS

Consider the following nonlinear system:

$$\Sigma : \left\{ \begin{array}{l} \dot{x} = f(x) + g(x)u + k(x)d \\ \|z\|^2 = \|h\|^2 + \|u\|^2 \end{array} \right\} \quad (1)$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ ,  $d \in \mathbb{R}^q$ ,  $f(x) \in \mathbb{R}^n$ ,  $g(x) \in \mathbb{R}^n \times \mathbb{R}^m$ ,  $k(x) \in \mathbb{R}^n \times \mathbb{R}^q$ ,  $h(x) \in \mathbb{R}^p$  with  $f(0) = 0$ , and  $x = 0$  is an equilibrium point of the system.  $z(t)$  is a fictitious output,  $d(t) \in L_2[0, \infty)$  is the disturbance, and  $u(t) \in U$  is the control with  $U$  defined as

$$U = \{u(t) \in L_2[0, \infty) \mid -\alpha_i \leq u_i \leq \alpha_i, i = 1, \dots, m\}.$$

The dynamics (1) is depicted in Fig. 1.

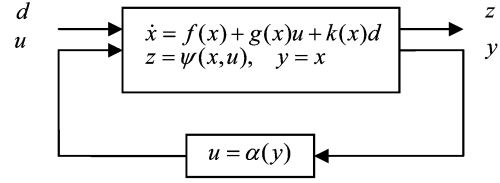


Fig. 1. State feedback nonlinear  $H_\infty$  controller.

In the  $L_2$ -gain problem, one is interested in  $u$ , which for some prescribed  $\gamma$  renders

$$V(x_0) = \int_0^\infty \left( \overbrace{\|z(t)\|^2}^{h'h + \|u\|^2} - \gamma^2 \|d\|^2 \right) dt \quad (2)$$

nonpositive for all  $d(t) \in L_2(0, \infty)$  with  $x(0) = 0$ . In other words

$$\int_0^\infty \|z(t)\|^2 dt \leq \gamma^2 \int_0^\infty \|d(t)\|^2 dt. \quad (3)$$

It is well known [6] that this problem is equivalent to the solvability of the zero-sum game

$$V^*(x_0) = \min_{u \in U} \max_d \int_0^\infty (h'h + \|u(t)\|^2 - \gamma^2 \|d\|^2) dt. \quad (4)$$

Note that this is a challenging constrained optimization since the minimization of the Hamiltonian with respect to  $u$  is constrained  $u \in U$ . In [2], to confront this constrained optimization problem, we use a quasi-norm to transform the constrained optimization problem (4) into

$$V^*(x_0) = \min_u \max_d \int_0^\infty (h'h + \|u\|_q^2 - \gamma^2 \|d\|^2) dt \quad (5)$$

where the minimization of the Hamiltonian with respect to  $u$  is unconstrained. See [1] and the work by Lyshevski [26] for similar work done in the framework of HJB equations.

A suitable quasi-norm to confront control saturation is given by

$$\|u\|_q^2 = 2 \int_0^u \phi^{-1}(v) dv = \sum_{k=1}^m 2 \int_0^{u_k} \phi^{-1}(v) dv \quad (6)$$

where  $\|u\|_q \in C^1$  is one to one, and  $\phi \in U$  with  $\phi$  being monotonically increasing, i.e.,  $\phi(\cdot) = \tanh(\cdot)$  for  $|u| \leq 1$ . Hence,  $\|u(t)\|_q^2 \simeq \|u(t)\|^2$  and is locally quadratic in  $u$ . Moreover, if  $\phi^{-1}$  is monotonically increasing, then

$$\int_b^a \phi^{-1}(v) dv - \phi^{-1}(b)'(a - b) > 0 \quad \forall a \neq b.$$

Substituting (6) in (5), the zero-sum game implies

$$\int_0^{\infty} \left( h'h + 2 \int_0^{u^*} \phi^{-1}(v) dv \right) dt \leq \gamma^2 \int_0^{\infty} \|d\|^2 dt. \quad (7)$$

From the dynamic programming principle 0, the minimax problem results in the following HJI equation:

$$\begin{aligned} V'_x f - V'_x g \phi \left( \frac{1}{2} g' V_x \right) + h'h + 2 \int_0^{-\phi(\frac{1}{2} g' V_x)} \phi^{-1}(v) dv \\ + \frac{1}{4\gamma^2} V'_x k k' V_x = 0, \quad V(0) = 0. \end{aligned} \quad (8)$$

According to the stationarity conditions, the optimal strategies are given as

$$u^*(x) = -\phi \left( \frac{1}{2} g' V_x \right) \quad d^*(x) = \frac{1}{2\gamma^2} k' V_x. \quad (9)$$

Note that the resulting control policy comes out naturally constrained due the use of the quasi-norm.

In [2], it is shown that one can iteratively solve for (8) by inner-loop iterations on  $d^i$  followed by outer-loop iterations on  $u_j$  in

$$\begin{aligned} V'_{x_j} (f_j + k d^i) + h'h + 2 \int_0^{u_j} \phi^{-T}(v) dv - \gamma^2 \|d^i\|^2 = 0 \\ f_j(x) = f(x) + g(x) u_j. \end{aligned} \quad (10)$$

Equation (10) is denoted as  $\text{PI}(V_j^i, u_j, d^i) = 0$ , where PI stands for policy iteration.

The inner- and outer-loop policy iterations go as follows. First, assume that one has a stabilizing controller  $u_j$ , then the inner-loop policy iterations

$$d^{i+1} = \frac{1}{2\gamma^2} k' V_x^i \quad (11)$$

will converge to the available storage  $V_j(x)$ , associated with  $u_j$ . Second, one updates the closed-loop dynamics by updating the constrained control policy according to

$$u_{j+1} = -\phi \left( \frac{1}{2} g' V_{x_j} \right) \quad (12)$$

where  $V_j$  is the available storage that solves

$$V'_{x_j} f_j + h'h + 2 \int_0^{u_j} \phi^{-1}(v) dv + \frac{1}{4\gamma^2} V'_{x_j} k k' V_{x_j} = 0.$$

In [2], it was shown that inner-loop iterations on  $i$  between (11) and (10) until convergence to the available storage  $V_j(x)$ , followed by an outer-loop iteration on  $j$ , controller update (12), converge to the value function of the HJI (8).

In the next section, it is demonstrated how to approximately solve for  $V_j^i$  in (10),  $\text{PI}(V_j^i, u_j, d^i) = 0$ , at each iteration on  $i$  and  $j$ .

### III. NEURAL NETWORK REPRESENTATION OF POLICIES

Although equation

$$V'_{x_j} (f_j + k d^i) + h'h + 2 \int_0^{u_j} \phi^{-T}(v) dv - \gamma^2 \|d^i\|^2 = 0 \quad (13)$$

is, in principle, easier to solve for  $V_j^i$  than solving the HJI (8) directly, it remains difficult to get an exact closed-form solution for  $V_j^i$  at each iteration. Therefore, one seeks to approximately solve for  $V_j^i$  at each iteration. In this section, a computationally practical neural-network-based algorithm is presented that solves for  $V_j^i$  on a compact set domain of the state space in a least squares sense. Proofs of convergence and stability of the neural network policies are discussed in Section IV.

It is well known that neural networks can be used to approximate smooth functions on prescribed compact sets [24]. Therefore,  $V_j^i$  is approximated at each inner-loop iteration  $i$  over a prescribed region of the state space with a neural net

$$\hat{V}_j^i(x) = \sum_{k=1}^L w_{j,k}^i \sigma_k(x) = \mathbf{w}_j^i \boldsymbol{\sigma}_L(x) \quad (14)$$

where the activation functions  $\sigma_j(x) : \Omega \rightarrow \mathfrak{R}$  are continuous  $\sigma_j 0 = 0$  span  $\{\sigma_j\}_1^\infty \subseteq L_2(\Omega)$ . The neural network weights are  $w_k$  and  $L$  is the number of hidden-layer neurons. Vectors

$$\begin{aligned} \boldsymbol{\sigma}_L(x) &\equiv [\sigma_1(x) \sigma_2(x) \cdots \sigma_L(x)]^T \\ \mathbf{w} &\equiv [w_1 w_2 \cdots w_L]^T \end{aligned}$$

are the vector activation function and the vector weight, respectively. The neural network weights are tuned to minimize the residual error in a least squares sense over a set of points within the stability region  $\Omega$  of the initial stabilizing control. The least squares solution attains the lowest possible residual error with respect to the neural network weights.

Replacing  $V_j^i$  in  $\text{PI}(V_j^i, u_j, d^i) = 0$  with  $\hat{V}_j^i$ , one has

$$\text{PI} \left( \hat{V}_j^i(x) = \sum_{k=1}^L w_k \sigma_k(x), u_j, d^i \right) = e_L(x) \quad (15)$$

where  $e_L(x)$  is the residual error.

To find the least squares solution, the method of weighted residuals is used [15]. The weights  $\mathbf{w}_j^i$  are determined by projecting the residual error onto  $de_L(x)/d\mathbf{w}_j^i$  and setting the result to zero  $\forall x \in \Omega$  using the inner product, i.e.,

$$\left\langle \frac{de_L(x)}{d\mathbf{w}_j^i}, e_L(x) \right\rangle = 0 \quad (16)$$

where  $\langle f, g \rangle = \int_{\Omega} f g dx$  is a Lebesgue integral. After rearranging the resulting terms, one has

$$\begin{aligned} \mathbf{w}_j^i &= - \langle \nabla \boldsymbol{\sigma}_L F_j^i, \nabla \boldsymbol{\sigma}_L F_j^i \rangle^{-1} \cdot \langle H_j^i, \nabla \boldsymbol{\sigma}_L F_j^i \rangle \\ F_j^i &= f + g u_j + k d^i \\ H_j^i &= h'h + 2 \int_0^{u_j} \phi^{-1}(v) dv - \gamma^2 \|d^i\|^2. \end{aligned} \quad (17)$$

Equation (17) involves a matrix inversion. The following lemma discusses the invertibility of this matrix.

*Lemma 1:* If the set  $\{\sigma_j\}_1^L$  is linearly independent, then

$$\{\nabla\sigma_j^T F_j^i\}_1^L$$

is also linearly independent.

*Proof:* This follows from the asymptotic stability of the vector field  $\dot{x} = F_j^i$  shown in [2], and from [1]. ■

Because of Lemma 1, the term  $\langle \nabla\sigma_L F_j^i, \nabla\sigma_L F_j^i \rangle$  is guaranteed to have a full rank, and thus is invertible, as long as  $\dot{x} = F_j^i$  is asymptotically stable. This in turn guarantees a unique  $\mathbf{w}_j^i$  of (17). Having solved for the neural net weights, the disturbance policy is updated as

$$\hat{d}^{i+1} = \frac{1}{2\gamma^2} k' \nabla\sigma_L^T \mathbf{w}_j^i. \quad (18)$$

It is important for the new dynamics  $\dot{x} = f + gu_j + k\hat{d}^{i+1}$  to be asymptotically stable in order to be able to solve for  $\mathbf{w}_j^{i+1}$  in (17). Theorem 1 discusses the asymptotic stability of  $\dot{x} = f + gu_j + k\hat{d}^{i+1}$ .

Policy iteration on the disturbance requires solving iteratively between (17) and (18) at each inner-loop iteration on  $i$  until the sequence of neural network weights  $\mathbf{w}_j^i$  converges to some value denoted by  $\mathbf{w}_j^*$ . Then, the control is updated using  $\mathbf{w}_j^*$  as

$$\hat{u}_{j+1} = -\phi \left( \frac{1}{2} g' \nabla\sigma_L^T \mathbf{w}_j^* \right) \quad (19)$$

in the outer-loop iteration on  $j$ .

Finally, one can approximate the integrals needed to solve (17) by introducing a mesh on  $\Omega$  with mesh size equal to  $\Delta x$ . Equation (17) becomes

$$\begin{aligned} X_j^i &= \left[ \nabla\sigma_L F_j^i|_{x_1} \cdots \cdots \nabla\sigma_L F_j^i|_{x_p} \right]^T \\ Y_j^i &= \left[ H_j^i|_{x_1} \cdots \cdots H_j^i|_{x_p} \right]^T \end{aligned} \quad (20)$$

where  $p$  in  $x_p$  represents the number of points of the mesh and  $H$  and  $F$  are as shown in (17). The number  $p$  increases as the mesh size is reduced. Therefore

$$\begin{aligned} \langle \nabla\sigma_L F_j^i, \nabla\sigma_L F_j^i \rangle &= \lim_{\|\Delta x\| \rightarrow 0} (X_j^i X_j^i) \cdot \Delta x \\ \langle H_j^i, \nabla\sigma_L F_j^i \rangle &= \lim_{\|\Delta x\| \rightarrow 0} (X_j^i Y_j^i) \cdot \Delta x. \end{aligned} \quad (21)$$

This implies that we can calculate  $\mathbf{w}_j^i$  as

$$\mathbf{w}_j^i = - \left( X_j^i X_j^i \right)^{-1} \left( X_j^i Y_j^i \right). \quad (22)$$

An interesting observation is that (22) is the standard least squares method of estimation for a mesh on  $\Omega$ . Note that the mesh size  $\Delta$  should be such that the number of points  $p$  is greater than or equal to the order of approximation  $L$ . This guarantees a full rank for  $(X_j^i X_j^i)$ .

There do exist various ways to efficiently approximate integrals as those appearing in (17). Monte Carlo integration techniques can be used. Here, the mesh points are sampled stochastically instead of being selected in a deterministic fashion [14]. In any case, however, the numerical algorithm at the end requires solving (22), which is a least squares computation of the

neural network weights. Numerically stable routines to compute equations like (22) do exist in several software packages like MATLAB which is used the next section.

A flowchart of the computational algorithm presented in this paper is shown in Fig. 2. This is an offline algorithm run *a priori* to obtain a neural network constrained state feedback controller that is nearly  $L_2$ -gain optimal. In this algorithm, once the policies converge for some  $\gamma_1$ , one may use the control policy as an initial policy for new inner–outer-loop policy iterations with  $\gamma_2 < \gamma_1$ . The attenuation  $\gamma$  is reduced until the HJI equation is no longer solvable on the desired compact set.

#### IV. STABILITY AND CONVERGENCE OF LEAST SQUARES NEURAL NETWORK POLICY ITERATIONS

In this section, the stability and convergence of policy iterations between (17), (18), and (19) is studied. Mainly, it is shown that the closed-loop dynamics resulting from the inner-loop iterations on the disturbance (18) is asymptotically stable as  $\hat{d}^{i+1}$  uniformly converges to  $d^{i+1}$ . Then later, we show that the updated  $\hat{u}_{j+1}$  is also stabilizing. Hence, this section starts by showing convergence results of the method of least squares when neural networks are used to solve for  $V_j^i$ . Note that (14) is a Fourier series expansion.

In this paper, we use a linear in parameter Volterra neural network. This gives a power series neural network that has the important property of being differentiable. This means that they can approximate uniformly a continuous function with all its partial derivatives up to order  $m$  using the same polynomial, by differentiating the series termwise. This type of series is  $m$ -uniformly dense as shown in [1]. Other  $m$ -uniformly dense neural networks, not necessarily based on power series, are studied in [17]. To study the convergence properties of the developed neural network algorithm, the following assumptions are required.

*Assumption 1:* It is assumed that the available storage exists and is positive definite. This is guaranteed for stabilizable dynamics and when the performance functional satisfies zero-state observability.

*Assumption 2:* The system dynamics and the performance integrands are such that the solution of the PI  $(V_j^i, u_j, d^i) = 0$  is continuous and differentiable for all  $i$  and  $j$ , therefore, belonging to the Sobolev space  $V \in H^{1,2}(\Omega)$ , [3].

*Assumption 3:* We can choose complete coordinate elements  $\{\sigma_j\}_1^\infty \in H^{1,2}(\Omega)$  such that the solution  $V \in H^{1,2}(\Omega)$  and  $\{\partial V / \partial x_1, \dots, \partial V / \partial x_n\}$  can be uniformly approximated by the infinite series built from  $\{\sigma_j\}_1^\infty$ .

*Assumption 4:* The sequence  $\{\psi_j = A\sigma_j\}$  is linearly independent and complete and given by

$$A\sigma_j = \frac{d\sigma_j'}{dx} (f + gu + kd).$$

Assumptions 1–3 are standard in  $H_\infty$  control theory and neural network control literature. Lemma 1 assures the linear independence required in the fourth assumption while the high-order Weierstrass approximation theorem [1], [17] shows that

$$\forall V, \varepsilon \exists L, \mathbf{w}_L : |\hat{V} - V| < \varepsilon \quad \forall k |d\hat{V}/dx_k - dV/dx_k| < \varepsilon$$

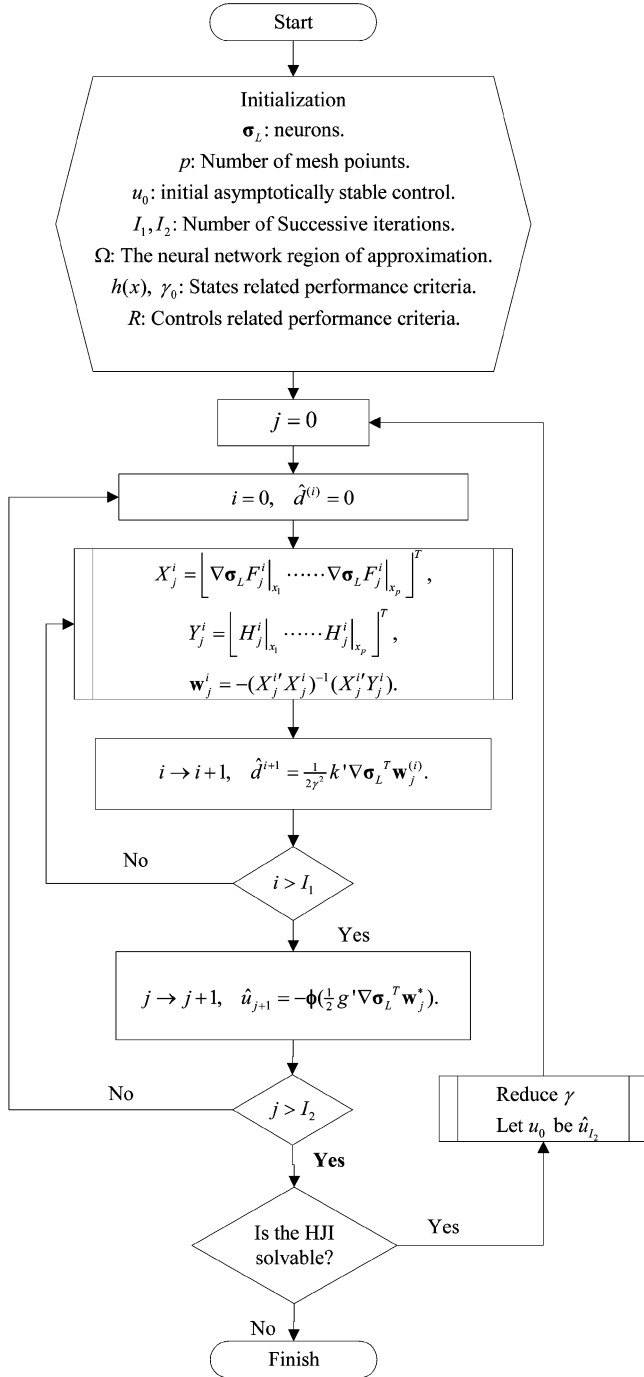


Fig. 2. Flowchart of the algorithm.

which implies that as  $L \rightarrow \infty$

$$\begin{aligned} \sup_{x \in \Omega} |A\hat{V} - AV| &\rightarrow 0 \\ \Rightarrow \|\hat{A}\hat{V} - AV\|_{L_2(\Omega)} &\rightarrow 0 \end{aligned}$$

and therefore, completeness of  $\{\psi_j\}$  is established, and Assumption 4 is satisfied.

Similar to the HJB equation [1], one can use the previous assumptions to conclude the uniform convergence of the least squares method which is placed in the Sobolev space  $H^{1,2}(\Omega)$ , [3].

**Theorem 1:** The neural network least squares approach converges uniformly for

$$\begin{aligned} \sup_{x \in \Omega} \left| d\hat{V}_j^i / dx - dV_j^i / dx \right| &\rightarrow 0 \\ \sup_{x \in \Omega} |\hat{V}_j^i - V_j^i| &\rightarrow 0 \\ \sup_{x \in \Omega} |\hat{d}^{i+1} - d^{i+1}| &\rightarrow 0 \\ \sup_{x \in \Omega} |\hat{u}_{j+1} - u_{j+1}| &\rightarrow 0. \end{aligned}$$

Next, it is shown that the system  $\dot{x} = f_j + k\hat{d}^{i+1}$  is asymptotically stable, and hence, (17) can be used to find  $\hat{V}^{i+1}$ .

**Theorem 2:**  $\exists L_0 : L \geq L_0$  such that  $\dot{x} = f_j + k\hat{d}^{i+1}$  is asymptotically stable.

*Proof:* Because the system  $\dot{x} = f_j + kd$  is dissipative with respect to  $\gamma$ , this implies [39] that there exists  $P(x) > 0$  such that

$$P_x' f_j + h'h + 2 \int_0^{u_j} \phi^{-1}(v) dv + \frac{1}{4\gamma^2} P_x' k k' P_x = Q(x) < 0 \quad (23)$$

where  $\forall i, P(x) \geq V^i(x)$ . Because

$$\begin{aligned} V_x^{i+1} \left( f_j + \frac{1}{2\gamma^2} k k' V_x^i \right) &= -h'h - 2 \int_0^{u_j} \phi^{-1}(v) dv \\ &\quad + \frac{1}{4\gamma^2} V_x^i k k' V_x^i \end{aligned} \quad (24)$$

one can write the following using (24) and (23):

$$\begin{aligned} (P_x - V_x^{i+1})' (f_j + k\hat{d}^{i+1}) &= P_x' f_j + h'h + 2 \int_0^{u_j} \phi^{-1}(v) dv + \frac{1}{4\gamma^2} P_x' k k' P_x \\ &\quad - \frac{1}{4\gamma^2} (P_x - V_x^i)' k k' (P_x - V_x^i) \\ &= Q(x) - \frac{1}{4\gamma^2} (P_x - V_x^i)' k k' (P_x - V_x^i) \\ &< 0. \end{aligned} \quad (25)$$

Because  $\dot{x} = f_j + k\hat{d}^{i+1}$  and the right-hand side of (25) is negative definite, it follows that  $P(x) - V^{i+1}(x) > 0$ . Using  $P(x) - V^{i+1}(x) > 0$  as a Lyapunov function candidate for the dynamics  $\dot{x} = f_j + k\hat{d}^{i+1}$ , one has

$$\begin{aligned} (P_x - V_x^{i+1})' \left( f_j + \frac{1}{2\gamma^2} k k' \hat{V}_x^i \right) &= P_x' f_j + h'h + 2 \int_0^{u_j} \phi^{-1}(v) dv + \frac{1}{4\gamma^2} P_x' k k' P_x \\ &\quad - \frac{1}{4\gamma^2} (P_x - V_x^i)' k k' (P_x - V_x^i) \\ &\quad + \frac{1}{2\gamma^2} (P_x - V_x^{i+1})' k k' (\hat{V}_x^i - V_x^i) \\ &\leq Q(x) + \frac{1}{2\gamma^2} (P_x - V_x^{i+1})' k k' (\hat{V}_x^i - V_x^i). \end{aligned}$$

From uniform convergence of  $\hat{V}^i$  to  $V^i, \exists L_0 : L \geq L_0$  such that

$$\forall x \in \Omega, \quad \frac{1}{2\gamma^2} (P_x - V_x^{i+1})' k k' (\hat{V}_x^i - V_x^i) > Q(x).$$

This implies that

$$\forall x \in \Omega, \quad (P_x - V_x^{i+1})' \left( f_j + \frac{1}{2\gamma^2} k k' \hat{V}_x^i \right) < 0. \quad \blacksquare$$

Next, it is shown that neural network policy iterations on the control as given by (19) is asymptotically stabilizing and  $L_2$ -gain stable for the same attenuation  $\gamma$  on  $\Omega$ .

*Theorem 3:*  $\exists L_0 : L \geq L_0$  such that  $\dot{x} = f + \hat{u}_{j+1}$  is asymptotically stable.

*Proof:* This proof is in essence contained in [1, Corollary 3], where the positive definiteness of  $h(x)$  is utilized by the fact that uniform convergence of  $\hat{V}_j$  to  $V_j$  implies that  $\exists L_0 : L \geq L_0$  such that

$$\forall x \in \Omega, \quad (V_{x_j})'(f + \hat{u}_{j+1}) < 0. \quad \blacksquare$$

*Theorem 4:* If  $\dot{x} = f + u_{j+1} + kd$  has  $L_2$ -gain less than  $\gamma$ , then  $\exists L_0 : L \geq L_0$  such that  $\dot{x} = f + \hat{u}_{j+1} + kd$  has  $L_2$ -gain less than  $\gamma$ .

*Proof:* Because  $\dot{x} = f + u_{j+1} + kd$  has  $L_2$ -gain less than  $\gamma$ , then this implies that there exists a  $P(x) > 0$  such that

$$P'_x(f + u_{j+1}) + h'h + 2 \int_0^{u_{j+1}} \phi^{-1}(v) dv + \frac{1}{4\gamma^2} P'_x k k' P_x = Q(x) < 0.$$

Hence

$$\begin{aligned} P'_x(f + \hat{u}_{j+1}) + h'h + 2 \int_0^{\hat{u}_{j+1}} \phi^{-1}(v) dv + \frac{1}{4\gamma^2} P'_x k k' P_x \\ = Q(x) + P'_x(\hat{u}_{j+1} - u_{j+1}) + 2 \int_{u_{j+1}}^{\hat{u}_{j+1}} \phi^{-1}(v) dv. \end{aligned}$$

From uniform convergence of  $\hat{u}_{j+1}$  to  $u_{j+1}$ ,  $\exists L_0 : L \geq L_0$  such that

$$\forall x \in \Omega, \quad P'_x(\hat{u}_{j+1} - u_{j+1}) + 2 \int_{u_{j+1}}^{\hat{u}_{j+1}} \phi^{-1}(v) dv > Q(x).$$

This implies that

$$\forall x \in \Omega, \quad P'_x(f + \hat{u}_{j+1}) + h'h + 2 \int_0^{\hat{u}_{j+1}} \phi^{-1}(v) dv + \frac{1}{4\gamma^2} P'_x k k' P_x < 0. \quad \blacksquare$$

The importance of Theorem 4 is that it justifies solving for the available storage for the new updated dynamics  $\dot{x} = f + \hat{u}_{j+1} + kd$ . Hence, all of the preceding theorems can be used to show by induction the following main convergence results.

*Theorem 5:*  $\exists L_0 : L \geq L_0$  such that the following hold.

- 1) For all  $j$ ,  $\dot{x} = f + \hat{u}_{j+1} + kd$  is dissipative with  $L_2$ -gain less than  $\gamma$  on  $\Omega$ .
- 2) For all  $j$  and  $i$ ,  $\dot{x} = f + \hat{u}_{j+1} + kd^i$  is asymptotically stable on  $\Omega$ .
- 3)  $\forall \varepsilon, \exists L_1 > L_0$  such that  $\sup_{x \in \Omega} |u_L^{(i)} - u^*| < \varepsilon$  and  $\sup_{x \in \Omega} |\hat{V}_j^i - V^*| < \varepsilon$ .

*Proof:* The proof follows directly from Theorems 1–4 by induction.  $\blacksquare$

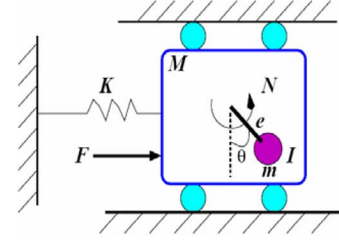


Fig. 3. Rotational actuator to control a translational oscillator.

## V. RTAC: THE NONLINEAR BENCHMARK PROBLEM

The RTAC benchmark problem was originally proposed in [10], and it has received much attention since then. The dynamics of this nonlinear plant poses a challenge as both the rotational and translation motions are coupled as shown. In [38] and [13], unconstrained controls were obtained to solve the  $L_2$  disturbance problem of the RTAC system based on Taylor series solutions of the HJI equation. In [28], unconstrained controllers based on the state-dependent Riccati equation (SDRE) were obtained. The SDRE is easier to solve than the HJI equation and results in a time-varying controller that was shown to be suboptimal.

In this section, a neural network constrained input  $H_\infty$  state feedback controller is computed for the RTAC shown in Fig. 3. To our knowledge, this is the first treatment in which inputs constraints are explicitly considered during the design of the optimal  $H_\infty$  controller that guarantees optimal disturbance attenuation.

The dynamics of the nonlinear plant is given as

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + k(x)d, & |u| &\leq 2 \\ z'z &= x_1^2 + 0.1x_2^2 + 0.1x_3^2 + 0.1x_4^2 + \|u\|_q^2 \\ \varepsilon &\triangleq me/\sqrt{(I + me^2)(M + m)} = 0.2, & \gamma &= 10 \end{aligned}$$

$$\begin{aligned} f &= \begin{bmatrix} x_2 \\ \frac{-x_1 + \varepsilon x_4^2 \sin x_3}{1 - \varepsilon^2 \cos^2 x_3} \\ x_4 \\ \frac{\varepsilon \cos x_3 (x_1 - \varepsilon x_4^2 \sin x_3)}{1 - \varepsilon^2 \cos^2 x_3} \end{bmatrix} \\ g &= \begin{bmatrix} 0 \\ \frac{-\varepsilon \cos x_3}{1 - \varepsilon^2 \cos^2 x_3} \\ 0 \\ \frac{1}{1 - \varepsilon^2 \cos^2 x_3} \end{bmatrix} \\ k &= \begin{bmatrix} 0 \\ \frac{1}{1 - \varepsilon^2 \cos^2 x_3} \\ 0 \\ \frac{-\varepsilon \cos x_3}{1 - \varepsilon^2 \cos^2 x_3} \end{bmatrix} \end{aligned} \quad (26)$$

with the states  $x_1 = r, x_2 = \dot{r}, x_3 = \theta$ , and  $x_4 = \dot{\theta}$  [10].

The design steps procedure goes as follows.

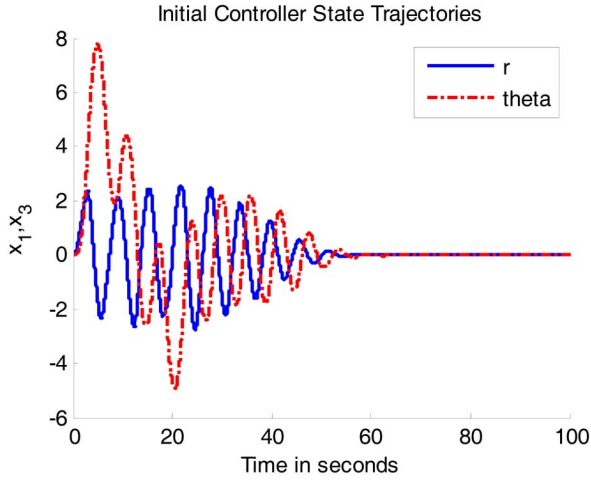


Fig. 4. The  $r, \theta$  state trajectories.

**A. Initial Control Selection**

The following  $H_\infty$  controller of the linear system resulting from Jacobian linearization of (26) is chosen

$$u_0 = 2 \tanh(2.4182x_1 + 1.1650x_2 - 0.3416x_3 - 1.0867x_4)$$

and forced to obey the  $|u| \leq 2$  constraint. This is a stabilizing controller that guarantees that  $L_2$ -gain  $< 6$  for the Jacobian linearized system [38]. The neural network is going to be trained on the following region of the state space  $|x_i| \leq 2 \quad i = 1, 2, 3, 4$ , which is a subset of the region of asymptotic stability of  $u_0$  that can be estimated using techniques in [10].

**B. Policy Iterations**

The iterative algorithm starts by approximately solving for the HJI with  $\gamma = 30$ . The approximate solution is done by inner-loop iterations between (18) and (22) followed by outer-loop policy iterations (19).

In the simulation performed, the neurons of the neural network were chosen from the sixth-order series expansion of the value function. Only polynomial terms of even order were considered, therefore having the total number of neural networks is  $L = 129$ . A sixth-order series approximation of the value function was satisfactory for our purposes, and it results in a fifth-order controller as done for the unconstrained case in [18]

$$\sigma_L = [x_1^2, x_1 x_2, x_1 x_3, x_1 x_4, x_2^2, x_2 x_3, x_2 x_4, x_3^2, x_3 x_4, x_4^2, x_1^4, x_1^3 x_2, x_1^3 x_3, x_1^3 x_4, x_1^2 x_2^2, x_1^2 x_2 x_3, x_1^2 x_2 x_4, x_1^2 x_3^2, x_1^2 x_3 x_4, x_1^2 x_4^2, x_1 x_2^3, x_1 x_2^2 x_3, x_1 x_2^2 x_4, x_1 x_2 x_3^2, x_1 x_2 x_3 x_4, x_1 x_2 x_4^2, x_1 x_3^3, x_1 x_3^2 x_4, x_1 x_3 x_4^2, x_1 x_4^3, x_2^4, x_2^3 x_3, x_2^3 x_4, x_2^2 x_3^2, x_2^2 x_3 x_4, x_2^2 x_4^2, x_2 x_3^3, x_2 x_3^2 x_4, x_2 x_3 x_4^2, x_2 x_4^3, x_3^4, x_3^3 x_4, x_3^2 x_4^2, x_3 x_4^3, x_4^4, x_1^6, x_1^5 x_2, x_1^5 x_3, x_1^5 x_4, x_1^4 x_2^2, x_1^4 x_2 x_3, x_1^4 x_2 x_4, x_1^4 x_3^2, x_1^4 x_3 x_4, x_1^4 x_4^2, x_1^3 x_2^3, x_1^3 x_2^2 x_3, x_1^3 x_2^2 x_4, x_1^3 x_2 x_3^2, x_1^3 x_2 x_3 x_4, x_1^3 x_2 x_4^2, x_1^3 x_3^3, x_1^3 x_3^2 x_4, x_1^3 x_3 x_4^2, x_1^3 x_4^3, x_1^2 x_2^4, x_1^2 x_2^3 x_3, x_1^2 x_2^3 x_4, x_1^2 x_2^2 x_3^2, x_1^2 x_2^2 x_3 x_4, x_1^2 x_2^2 x_4^2, x_1^2 x_2 x_3^3,$$

$$x_1^2 x_2 x_3^2 x_4, x_1^2 x_2 x_3 x_4^2, x_1^2 x_2 x_4^3, x_1^2 x_3^4, x_1^2 x_3^3 x_4, x_1^2 x_3^2 x_4^2, x_1^2 x_3 x_4^3, x_1^2 x_4^4, x_1 x_2^5, x_1 x_2^4 x_3, x_1 x_2^4 x_4, x_1 x_2^3 x_3^2, x_1 x_2^3 x_3 x_4, x_1 x_2^3 x_4^2, x_1 x_2^2 x_3^3, x_1 x_2^2 x_3^2 x_4, x_1 x_2^2 x_3 x_4^2, x_1 x_2^2 x_4^3, x_1 x_2 x_3^4, x_1 x_2 x_3^3 x_4, x_1 x_2 x_3^2 x_4^2, x_1 x_2 x_3 x_4^3, x_1 x_2 x_4^4, x_1 x_3^5, x_1 x_3^4 x_4, x_1 x_3^3 x_4^2, x_1 x_3^2 x_4^3, x_1 x_3 x_4^4, x_1 x_4^5, x_2^6, x_2^5 x_3, x_2^5 x_4, x_2^4 x_3^2, x_2^4 x_3 x_4, x_2^4 x_4^2, x_2^3 x_3^3, x_2^3 x_3^2 x_4, x_2^3 x_3 x_4^2, x_2^3 x_4^3, x_2^2 x_3^4, x_2^2 x_3^3 x_4, x_2^2 x_3^2 x_4^2, x_2^2 x_3 x_4^3, x_2^2 x_4^4, x_2 x_3^5, x_2 x_3^4 x_4, x_2 x_3^3 x_4^2, x_2 x_3^2 x_4^3, x_2 x_3 x_4^4, x_2 x_4^5, x_3^6, x_3^5 x_4, x_3^4 x_4^2, x_3^3 x_4^3, x_3^2 x_4^4, x_3 x_4^5, x_4^6 E].$$

Once the neural network algorithm converges, and an approximate solution for (8) with  $\gamma = 30$ , the resulting controller can be used as an initial controller for a new inner–outer-loop iterations to solve (8) with a smaller  $\gamma$ .

The computational routine was successful in obtaining approximate solutions to (8) with  $\gamma = 10$  with the final weights given as

$$\mathbf{w} = [7.5591 - 0.5592 - 0.0398 - 2.0616 \ 7.5212 \ 1.7514 \dots 3.0072 \ 0.3526 \ 1.2436 \ 1.3561 \ 0.0910 \ 0.0082 \dots - 0.1817 - 0.1380 \ 0.1958 \ 0.1807 \ 0.1441 \ 0.3113 \dots 0.4315 \ 0.2912 \ 0.0057 - 0.1288 - 0.0817 \ 0.2979 \dots 0.3864 \ 0.1383 - 0.2192 \ 0.4320 \ 0.1636 \ 0.0131 \dots 0.1107 \ 0.1727 \ 0.2055 \ 0.0897 \ 0.3292 \ 0.3234 \dots - 0.4341 - 1.9855 - 0.1703 - 0.0064 \ 0.1540 - 0.1364 \dots - 0.2915 \ 0.0053 \ 0.0407 \ 0.0029 - 0.0125 \ 0.0142 \dots 0.0071 \ 0.0061 - 0.0099 - 0.0072 - 0.0060 - 0.0123 \dots - 0.0082 - 0.0110 \ 0.0289 \ 0.0193 \ 0.0033 - 0.0147 \dots 0.0052 \ 0.0074 \ 0.0098 \ 0.0001 \ 0.0016 \ 0.0047 \dots - 0.0138 - 0.0084 - 0.0047 - 0.0192 - 0.0258 - 0.0177 \dots - 0.0408 - 0.0187 - 0.0053 - 0.0012 - 0.0144 - 0.0260 \dots - 0.0080 \ 0.0062 - 0.0011 \ 0.0140 \ 0.0109 - 0.0031 \dots - 0.0127 - 0.0051 - 0.0041 - 0.0134 - 0.0131 - 0.0141 \dots - 0.0292 - 0.0178 - 0.0089 - 0.0243 - 0.0125 \ 0.0022 \dots - 0.0482 - 0.0388 \ 0.0184 \ 0.0366 \ 0.0064 \ 0.0011 \dots - 0.0063 - 0.0042 - 0.0004 - 0.0102 - 0.0150 - 0.0141 \dots - 0.0515 - 0.0319 - 0.0144 \ 0.0157 \ 0.0003 \ 0.0200 \dots 0.0398 \ 0.0091 \ 0.0346 \ 0.1461 - 0.0217 - 0.0407 \dots - 0.0048 - 0.0008 - 0.0273 \ 0.0100 \ 0.0493 \ 0.0037 \dots - 0.0105 - 0.0167 - 0.0058]^T.$$

The controller is finally given as

$$u = - \tanh \left( \frac{1}{2} g^T(x) \nabla \sigma_L^T \mathbf{w} \right).$$

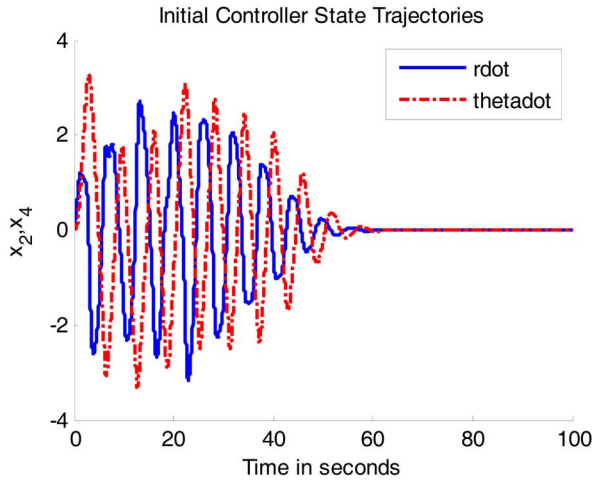


Fig. 5. The  $\dot{r}, \dot{\theta}$  state trajectories.

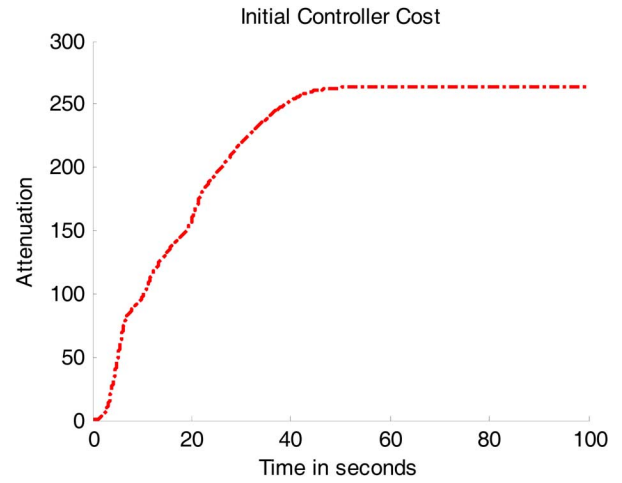


Fig. 7. Disturbance attenuation.

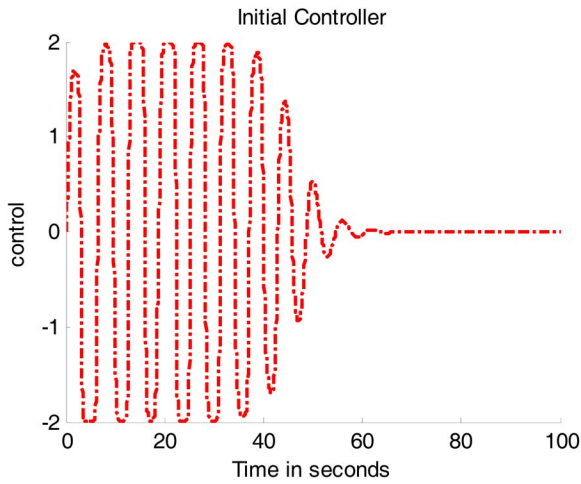


Fig. 6. The  $u(t)$  control input.

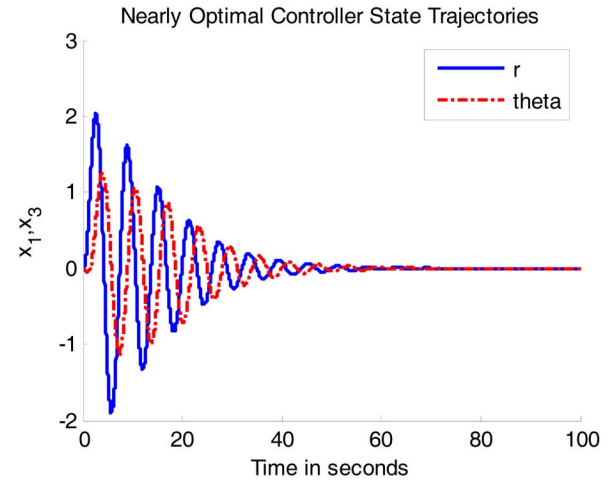


Fig. 8. The  $r, \theta$  state trajectories.

C. Simulation

Figs. 4 and 5 show the states trajectories when the system is at rest and experiencing a disturbance  $d(t) = 5 \sin(t)e^{-t}$ . Fig. 6 shows the control signal, while Fig. 7 shows the attenuation

$$\int_0^\infty \|z(t)^T z(t)\|^2 dt \Big/ \int_0^\infty \|d(t)\|^2 dt.$$

Figs. 8 and 9 show the states trajectories when the system is at rest and experiencing a disturbance  $d(t) = 5 \sin(t)e^{-t}$ . Figs. 10 and 11 show the control signal and attenuation, respectively.

The nearly optimal nonlinear constrained input  $H_\infty$  controller is shown to perform much better than the initial controller the algorithm started with. The algorithm presented in this paper allowed a novel utilization of neural networks approximation property to obtain closed-form solution to the constrained input  $H_\infty$  control policy that is impossible using existing approaches in the literature.

Note that the activation function may be chosen to be any-thing satisfying the four assumptions previously discussed in

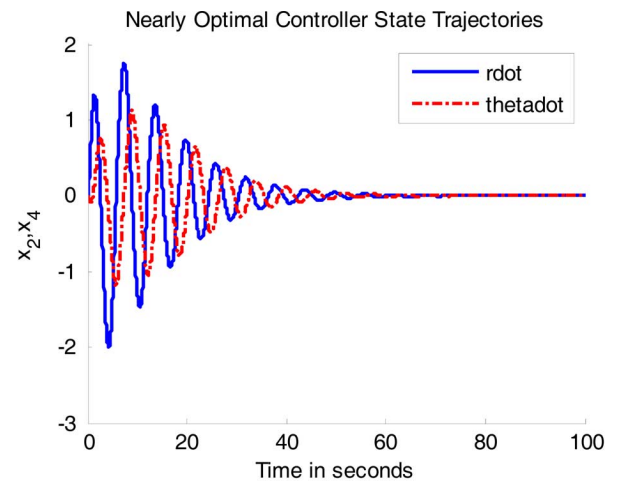


Fig. 9. The  $\dot{r}, \dot{\theta}$  state trajectories.

Section IV. This includes but is not limited to, sigmoid functions, radial basis functions, and others. In this simulation, we have chosen polynomial functions only to compare our results to those in [18] and [38].



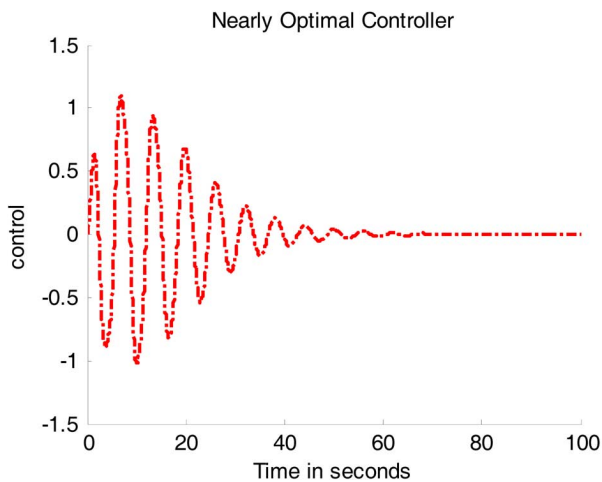


Fig. 10. The  $u(t)$  control input.

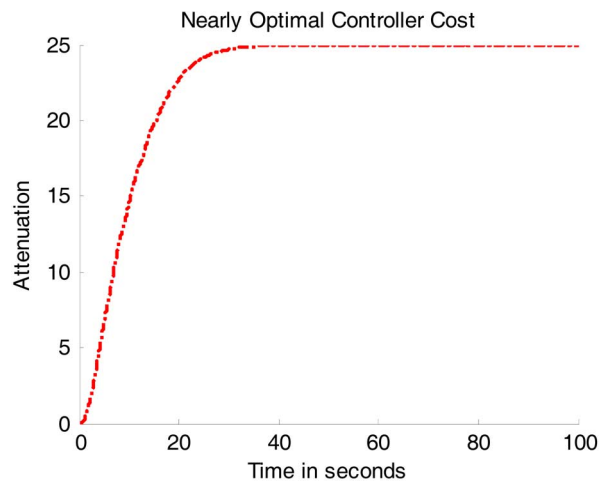


Fig. 11. Disturbance attenuation.

## VI. CONCLUSION

This paper presents an application of neural networks to find closed-form representation of feedback strategies for a zero-sum game that appears in the  $H_\infty$  control. The systems considered are affine in input with control saturation. The algorithm relies on policy iterations that has been proposed for unconstrained [8] and constrained [2] control case. The presented algorithm is an extension to the optimal quadratic regulations for constrained inputs using the HJB equation appearing in [1]. The results of this paper and 0 can be further researched to provide an adaptive optimal control schemes, approximate dynamic programming, in which the presented algorithm is required to be implemented online.

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