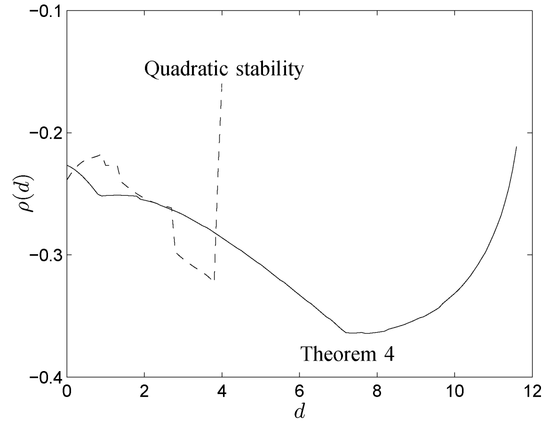

Fig. 1. Degree of stability.

gain K. However, for d slightly greater than four the quadratic stability constraints become infeasible. In the same figure, the solid line shows that the conditions provided by Theorem 4 are always feasible for all $0 \leq d \leq 11.62$. Hence, in this example, the stability conditions provided by Theorem 4 perform significantly better than the quadratic stability conditions. For $d = 11.62$, the state feedback gain

$$K = [\,-17.9999 \quad -49.5487 \quad -1.9946 \quad -4.1078\,] \tag{31}$$

assures that $\mathcal{A}(K)$ is Hurwitz stable but this fact can not be tested by means of a common Lyapunov function.

VI. Conclusion

In this note, we have provided sufficient conditions to guarantee that a matrix quadratic polynomial with an arbitrary although finite number of variables has invariant sign in a simplex. We believe that the reported conditions are useful in matrix analysis and in particular on the determination of robust performance bounds for problems involving polytopic parameter uncertainties. We have shown theoretically that several sufficient robust stability conditions available in the literature to date are special cases of the proposed ones. In our opinion, one of the main contributions of this note is a new matrix gain parametrization which is used to design stabilizing state feedback controllers from the proposed stability conditions and which is sufficiently general to cope with other classes of robust control design problems. This aspect and others including discrete time systems stability, performance analysis and control design are now under investigation.

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Policy Iterations on the Hamilton–Jacobi–Isaacs Equation for H_∞ State Feedback Control With Input Saturation

Murad Abu-Khalaf, Frank L. Lewis, and Jie Huang

***Abstract*—An H_∞ suboptimal state feedback controller for constrained input systems is derived using the Hamilton–Jacobi–Isaacs (HJI) equation of a corresponding zero-sum game that uses a special quasi-norm to encode the constraints on the input. The unique saddle point in feedback strategy form is derived. Using policy iterations on both players, the HJI equation is broken into a sequence of differential equations linear in the cost for which closed-form solutions are easier to obtain. Policy iterations on the disturbance are shown to converge to the available storage function of the associated L_2-gain dissipative dynamics. The resulting constrained optimal control feedback strategy has the largest domain of validity within which L_2-performance for a given γ is guaranteed.**

***Index Terms*—Controller saturation, H_∞ control, policy iterations, zero-sum games.**

I. Introduction

In this note, we derive the Hamilton–Jacobi–Isaacs (HJI) equation for systems with input constraints and then develop an algorithm based on policy iterations to solve the obtained HJI equation. Although the formulation of the nonlinear H_∞ control theory has been well developed, [4], [5], [7], [11], [17], and [19], solving the corresponding HJI equation remains a challenge. Several methods have been proposed to solve the HJI equation. When its solution is smooth, it can be deter-

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M. Abu-Khalaf and F. L. Lewis are with the Automation and Robotics Research Institute, The University of Texas at Arlington, Fort Worth, TX 76118 USA (e-mail: abukhalaf@arri.uta.edu; lewis@uta.edu).

J. Huang is with the Department of Automation and Computer-Aided Engineering, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong (e-mail: jhuang@acae.cuhk.edu.hk).

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mined directly by solving for the coefficients of the Taylor series expansion of the value function, as it has been proposed in [10]. In [17], it was proven that there exist a sequence of policy iterations on the control input to pursue the smooth solution of the HJI equation. Later in [8], policy iterations on the disturbance input was suggested in addition to policy iterations on the control input. However, the existence and stability of the disturbance policy iterations were not proven.

In this note, we have three objectives. First, prove the existence of policy iterations on the disturbance input under certain assumptions and show their convergence to the available storage function of the associated dissipative closed-loop dynamics. Second, we give a formal solution to the suboptimal H_∞ control problem of dynamical systems with input constraints using a special quasi-norm to obtain a nonquadratic zero-sum game and derive the corresponding HJI equation. Third, policy iterations on both players, the control and disturbance inputs, are used to solve for the optimal strategies of the nonquadratic zero-sum game. The policy iterations method results in a sequence of linear partial differential equations whose solutions are shown to converge to the game value function that solves the HJI. Two scalar examples are presented to illustrate the theory.

A major contribution of this note is that the two-player policy iterations scheme generates equations that are easier to solve compared to the original HJI equation of the corresponding constrained input zero-sum game. In [2], a neural network solution method is presented to solve the linear partial differential equations resulting from the two-player policy iterations.

The two-player policy iterations scheme we present in this note is a significant improvement on our earlier work in [1] where one-player policy iterations is used to solve the HJB equation appearing in constrained input optimal control theory. The role of this note is to rigorously examine two-player policy iterations to zero-sum games that also have in addition saturation constraints. Together both papers rigorously demonstrate the role of policy iterations, a machine learning approximate dynamic programming scheme that is well established in computer science [13], to optimal control theory and zero-sum game theory.

Remark 1: Necessary conditions for the existence of smooth solutions of the HJI equation in the case of systems with no input constraints have been studied earlier by [11], [17]. Other lines of research study the nonsmooth solutions of the HJI equation using the theory of viscosity solutions, [5]. This notion of solutions was studied for the H_∞ control problem in [4]. The results in this note are done under regularity assumptions as done in [11], [17], and [1] for the HJB case.

II. Policy Iterations and the Available Storage $V_a(x)$

Consider the system described by

$$\begin{aligned} \dot{x} &= f(x) + k(x)d \\ z &= h(x) \end{aligned} \tag{1}$$

where $f(0) = 0$, $d(t)$ is a disturbance, and $z(t)$ is a fictitious output. $x = 0$ is assumed to be an equilibrium point of the system. It is said that (1) has an L_2-gain $\leq \gamma, \gamma \geq 0$, if

$$\int_0^T \|z(t)\|^2 \, dt \leq \gamma^2 \int_0^T \|d(t)\|^2 \, dt \tag{2}$$

for all $T \geq 0$ and all $d \in L_2(0,T)$, with $x(0) = 0$. Dynamical systems that are finite L_2-gain stable are said to be dissipative, [19]. The existence of the so-called *available storage* function is essential in determining whether or not a system is dissipative.

Definition 1: The available storage V_a when it exists is the solution of the optimal control problem

$$V_a(x) = \sup_{d(\cdot), T \geq 0} \int_0^T \left\{ \|z(t)\|^2 - \gamma^2 \|d(t)\|^2 \right\} dt.$$

When the available storage $V_a \geq 0$ is smooth $V_a \in C^1$ and $T \to \infty$, it solves the HJ equation

$$V_{a_x} f + \frac{1}{4\gamma^2} V a'_x k k' V a_x + h'h = 0 \quad V_a(0) = 0. \tag{3}$$

If in addition zero-state observability is assumed, then $V_a > 0$ and has a certain domain of validity.

Definition 2: The domain of validity (DOV) of V_a is the set Ω of all x satisfying (3), [9].

The next Lemma is taken from [14], [17] and is used later in the proof of Theorem 1.

Lemma 1: If (1) with $d = 0$ is asymptotically stable and in addition has an L_2- gain $< \gamma$, and if the available storage is smooth, then the closed-loop dynamics

$$\dot{x} = f + \frac{1}{2\gamma^2} k k' V_{a_x} \tag{4}$$

is asymptotically stable. Moreover, one can find $P(x) > 0$ and $\varepsilon(x) > 0$ such that

$$P'_x f + \frac{1}{4\gamma^2} P'_x k k' P_x + h'h + \varepsilon(x) = 0 \tag{5}$$

is satisfied locally around the origin.

Proof: See [14] and [17, eq. (85)]. ■

Equation (3) is nonlinear in $V_a(x)$, therefore in general it is hard if not impossible to solve. In Theorem 1, policy iterations on d are used to break (3) into a sequence of equations that are linear in $V(x)$. This type of policy iterations becomes Newton's method to solve the Riccati equation

$$A'P + PA + \frac{1}{\gamma^2} PKK'P + H'H = 0 \tag{6}$$

that appears in the Bounded Real Lemma problem for linear systems [15], [20].

Theorem 1: Let the system (1) be zero-state observable, locally asymptotically stable with $d = 0$, and in addition has an L_2- gain $< \gamma$. Assume that the available storage is a smooth function $V^* > 0 \in C^1$ with a DOV Ω^*. Then starting with $d^0 = 0$ and assuming that $\forall i \; V^i \in C^1$, there exists a sequence of policies resulting from iterations between (7) and (8)

$$V_x^{i'}(f + kd^i) + h'h - \gamma^2 \|d^i\|^2 = 0 \tag{7}$$

$$d^i = \frac{1}{2\gamma^2} k' V_x^{i-1} \tag{8}$$

such that $\dot{x} = f + kd^i$ is locally asymptotically stable $\forall i$. Moreover

$$i \to \infty \Rightarrow \sup_{x \in \Omega^*} |V^i - V^*| \to 0$$

with $0 < V^i(x) \leq V^{i+1}(x) \; \forall x \in \Omega^*$ and $\Omega^i \subseteq \Omega^{i-1}$.

Proof: Assume that there is d^i such that $\dot{x} = f + kd^i$ is asymptotically stable. Then

$$V^i(x_0) = \int_0^\infty \left\{ h'h - \frac{1}{4\gamma^2} V_x^{i-1\prime} kk' V_x^{i-1} \right\} dt \tag{9}$$

is well defined and its infinitesimal version is

$$V_x^{i\prime}\left(f + \frac{1}{2\gamma^2} kk' V_x^{i-1}\right) = -h'h + \frac{1}{4\gamma^2} V_x^{i-1\prime} kk' V_x^{i-1} \tag{10}$$

from which one may note that $\Omega^i \subseteq \Omega^{i-1}$. Adding and subtracting terms to (5) in Lemma 1, one has

$$\begin{aligned} P_x'\left(f + \frac{1}{2\gamma^2} kk' V_x^{i-1}\right) = &-h'h - \varepsilon(x) + \frac{1}{4\gamma^2} V_x^{i-1\prime} kk' V_x^{i-1} \\ &- \frac{1}{4\gamma^2}\left(P_x - V_x^{i-1}\right)' kk' \left(P_x - V_x^{i-1}\right). \end{aligned} \tag{11}$$

Combining (10) and (11), it follows that

$$\begin{aligned} &\left(P_x - V_x^i\right)'\left(f + \frac{1}{2\gamma^2} kk' V_x^{i-1}\right) \\ &\quad = -\varepsilon(x) - \frac{1}{4\gamma^2}\left(P_x - V_x^{i-1}\right)' kk' \left(P_x - V_x^{i-1}\right) < 0. \end{aligned} \tag{12}$$

Since the vector field $\dot{x} = f + kd^i$ is locally asymptotically stable, it follows that $P - V^i > 0$ is a Lyapunov function. To show local asymptotic stability of $\dot{x} = f + kd^{i+1}$, differentiating V^i over the trajectories of $\dot{x} = f + kd^{i+1}$ and noting (10), one has

$$\begin{aligned} V_x^{i\prime}\left(f + \frac{1}{2\gamma^2} kk' V_x^i\right) = &-h'h + \frac{1}{4\gamma^2} V_x^{i\prime} kk' V_x^i \\ &+ \frac{1}{4\gamma^2}\left(V_x^i - V_x^{i-1}\right)' kk' \left(V_x^i - V_x^{i-1}\right) \end{aligned} \tag{13}$$

and similar to (11), one has

$$\begin{aligned} P_x'\left(f + \frac{1}{2\gamma^2} kk' V_x^i\right) = &-h'h - \varepsilon(x) + \frac{1}{4\gamma^2} V_x^{i\prime} kk' V_x^i \\ &- \frac{1}{4\gamma^2}\left(P_x - V_x^i\right)' kk' \left(P_x - V_x^i\right). \end{aligned} \tag{14}$$

Using $P - V^i > 0$ as a Lyapunov function candidate and differentiating it along the trajectories of $\dot{x} = f + kd^{i+1}$, one obtains by combining (13) and (14)

$$\begin{aligned} &\left(P_x - V_x^i\right)'\left(f + \frac{1}{2\gamma^2} kk' V_x^i\right) \\ &= -\varepsilon(x) - \frac{1}{4\gamma^2}\left(P_x - V_x^i\right)' kk' \left(P_x - V_x^i\right) \\ &\quad - \frac{1}{4\gamma^2}\left(V_x^i - V_x^{i-1}\right)' kk' \left(V_x^i - V_x^{i-1}\right) \\ &< 0. \end{aligned}$$

Hence, $\dot{x} = f + kd^{i+1}$ is locally asymptotically stable. Starting with $d^0 \equiv 0$, and by asymptotic stability of $\dot{x} = f$, it follows by induction that $\dot{x} = f + kd^i$ is stable $\forall i$. To show uniform convergence of V^i to V^*, note that

$$\begin{aligned} V_x^{i+1\prime}(f + kd^{i+1}) &= -h'h + \gamma^2 \|d^{i+1}\|^2 \\ V_x^{i\prime} f &= -V_x^{i\prime} kd^i - h'h + \gamma^2 \|d^i\|^2 \\ V_x^{i\prime} k &= 2\gamma^2 d^{i+1\prime}. \end{aligned}$$

By integrating V^i and V^{i+1} over the state trajectory of $\dot{x} = f + kd^{i+1}$ for $x_0 \in \Omega^i \wedge \Omega^{i+1}$. It follows that

$$\begin{aligned} &V^{i+1}(x_0) - V^i(x_0) \\ &\quad = -\int_0^\infty \left\{ \dot{V}^{i+1}(x_0) - \dot{V}^i(x_0) \right\} dt \\ &\quad = \int_0^\infty \left\{ V_x^{i\prime}(f + kd^{i+1}) - V_x^{i+1\prime}(f + kd^{i+1}) \right\} dt \\ &\quad = \int_0^\infty \gamma^2 \left\{ \|d^i\|^2 + 2d^{i+1\prime}(d^{i+1} - d^i) - \|d^{i+1}\|^2 \right\} dt \\ &\quad = \gamma^2 \int_0^\infty \left\{ \|d^{i+1} - d^i\|^2 \right\} dt \ge 0 \end{aligned}$$

and hence pointwise convergence to the solution of (3) follows. Since Ω^* is compact, uniform convergence of V^i to V^* on Ω^* follows from Dini's theorem, [3]. Finally, zero-state observability guarantees that $V^0(x) > 0$. ■

Note that for linear systems, $\forall i$ $V^i(x)$ is a quadratic function.

Lemma 2: Let the L_2-gain of (1) be γ^* with $\gamma_1 \ge \gamma_2 > \gamma^*$. If the available storages $V_{\gamma_1} \in C^1$ and $V_{\gamma_2} \in C^1$ solve (3) with γ_1 and γ_2, respectively, then $V_{\gamma_2} \ge V_{\gamma_1}$ with $\Omega_{\gamma_2} \subseteq \Omega_{\gamma_1}$.

Proof: Since for γ_2, the available storage $V_{\gamma_2}^*$ satisfies

$$V_{x\gamma_2}^{*\prime} f + \frac{1}{4\gamma_2^2} V_{x\gamma_2}^{*\prime} kk' V_{x\gamma_2}^* + h'h = 0.$$

Replacing γ_2 with γ_1 one has

$$V_{x\gamma_2}^{*\prime} f + \frac{1}{4\gamma_1^2} V_{x\gamma_2}^{*\prime} kk' V_{x\gamma_2}^* + h'h \le 0.$$

$V_{\gamma_2}^*$ is now a possible storage function for (1) with gain γ_1. Therefore, $V_{\gamma_1}^* \le V_{\gamma_2}^*$ and $\Omega_{\gamma_2} \subseteq \Omega_{\gamma_1}$. ■

The following example illustrates the policy iterations theory to solve for the available storage.

Example 1: Consider the nonlinear system

$$\dot{x} = -x^3 + d \quad z = x^3. \tag{15}$$

The corresponding HJ equation is

$$V_x(-x^3) + \frac{1}{4\gamma^2} V_x^2 + x^6 = 0 \tag{16}$$

The available storage is $V(x) = 2\gamma^2(1 - (1 - \gamma^{-2})^{1/2})x^4/4$. Note that the available storage cease to exists for $\gamma < 1$. Hence the L_2-gain is equal to 1. Note that the closed-loop dynamics with $d = (1-(1-\gamma^{-2})^{1/2})x^3$ is

$$\dot{x} = -(1-\gamma^{-2})^{1/2}x^3 \tag{17}$$

and hence it is asymptotically stable for $\gamma > 1$.

To solve the HJ (16) by policy iterations, note that $V^i(x) = p_i x^4$ with p_i a constant. Hence, (7) gives

$$4p_i x^3\left(-x^3 + \frac{2}{\gamma^2}p_{i-1}x^3\right) + x^6 - \gamma^2\left\|\frac{2}{\gamma^2}p_{i-1}x^3\right\|^2 = 0 \tag{18}$$

which is equivalent to

$$p_i(-4+8\gamma^{-2}p_{i-1}) + 1 - 4\gamma^{-2}p_{i-1}^2 = 0. \tag{19}$$

For the case when $\gamma = 2$, $V(x) = (2-\sqrt{3})x^4$. Iterating on (19) with $p_0 = 0$ converges to $p_\infty = 2-\sqrt{3}$.

Note that for arbitrary $f(x)$, $k(x)$ and $h(x)$, the analytical solution to the HJ (3) is not possible in general. However, one may use techniques such as neural networks to obtain a closed-form approximation of the exact solution to (7) over a domain of the state–space [2].

III. L_2-Gain of Nonlinear Control Systems With Input Saturation

Consider the following nonlinear system:

$$\Sigma : \begin{Bmatrix} \dot{x} = f(x) + g(x)u + k(x)d, \\ \|z\|^2 = \|h\|^2 + \|u\|^2 \end{Bmatrix} \tag{20}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $d \in \mathbb{R}^q$, $f(0) = 0$, $x = 0$ is an equilibrium point of the system, $z(t)$ is a fictitious output, $d(t) \in L_2[0,\infty)$ is the disturbance, and $u(t) \in U$ is the control with U defined as

$$U = \{u(t) \in L_2[0,\infty) | -\alpha_i \le u_i \le \alpha_i, \qquad i = 1,\ldots,m\}.$$

In the L_2-gain problem, one is interested in u which for some prescribed γ and $x(0) = 0$ renders

$$\int_0^\infty \left(\overbrace{h'h + \|u\|^2}^{\|z(t)\|^2} - \gamma^2\|d\|^2\right) dt \tag{21}$$

nonpositive for all $d(t) \in L_2(0,\infty)$. In other words

$$\int_0^\infty \|z(t)\|^2\, dt \le \gamma^2 \int_0^\infty \|d(t)\|^2\, dt. \tag{22}$$

It is well known, [7], that this problem is equivalent to the solvability of the zero-sum game

$$V^*(x_0) = \min_{u\in U}\max_d \int_0^\infty \left(h'h + \|u(t)\|^2 - \gamma^2\|d\|^2\right) dt \tag{23}$$

Note that this is a challenging constrained optimization since the minimization of the Hamiltonian with respect to u is constrained, $u \in U$. To confront this constrained optimization problem, we propose the use of a quasi-norm to transform the constrained optimization problem (23) into

$$V^*(x_0) = \min_u \max_d \int_0^\infty \left(h'h + \|u\|_q^2 - \gamma^2\|d\|^2\right) dt \tag{24}$$

where applying the stationarity conditions for minimizing u becomes direct. In this case, $\|u\|_q^2$ is defined for $u \in U$. See [1] and [16] for similar work done in the framework of HJB equations.

Definition 3: A quasi-norm, $\|\cdot\|_q$, on a vector space X, has the following properties:

$$\|x\|_q = 0 \Leftrightarrow x = 0,\ \|x+y\|_q \le \|x\|_q + \|y\|_q,\ \|x\|_q = \|-x\|_q.$$

This definition is weaker than the definition of a norm, in which the third property is replaced by homogeneity, $\|\alpha x\|_q = |\alpha|\|x\|_q \forall\alpha \in \Re$, [3]. A suitable quasi-norm to confront input saturation is

$$\|u\|_q^2 = 2\int_0^u \phi^{-1}(v)dv = \sum_{k=1}^m 2\int_0^{u_k} \phi^{-1}(v)dv \tag{25}$$

where $\|u\|_q \in C^1$ is one to one and ϕ^{-1} is assumed to be monotonically increasing. This implies the following Lemma.

Lemma 3: Let ν belong to the domain of $\phi^{-1}(\nu)$, if $\phi^{-1}(\nu)$ is monotonically increasing in ν, then [1]

$$\int_b^a \phi^{-1}(v)dv - \phi^{-1}(b)'(a-b) > 0 \qquad \forall a \ne b.$$

An example is the use of $\phi(\cdot) = \tanh(\cdot)$ when $|u| \le 1$. In this case, the range of $\phi(\cdot)$ and the domain of $\phi^{-1}(u)$ is $(-1,1)$ and, therefore, satisfying the constraints.

Substituting (25) in (24) implies

$$\int_0^\infty \left(h'h + 2\int_0^{u^*} \phi^{-1}(v)dv\right) dt \le \gamma^2 \int_0^\infty \|d\|^2 dt. \tag{26}$$

IV. The HJI Equation and the Saddle Point

Equation (24) is a zero-sum game with feedback strategy information structure for both players, [7]. It is shown in Lemma 4 that Isaacs's condition is satisfied and there is a unique saddle point solving the finite-horizon zero-sum game

$$V^*(x_0,T) = \min_u \max_d \int_0^T \left(h'h + 2\int_0^u \phi^{-1}(v)dv - \gamma^2\|d\|^2\right) dt. \tag{27}$$

The Hamiltonian of the game (27) is

$$H(x,p,u,d)=p'(f+gu+kd)+h'h+2\int_0^u \phi^{-1}(v)dv-\gamma^2\|d\|^2. \quad (28)$$

From Lemma 3, Isaacs's condition follows as shown in the next Lemma.

Lemma 4: For the Hamiltonian (28), Isaacs's condition is satisfied $\min_u \max_d H = \max_d \min_u H$.

Proof: Applying the stationarity conditions $\partial H/\partial u = 0$, $\partial H/\partial d = 0$ on (28) gives (29)

$$2\phi^{-1}(u^*)+g(x)'p=0 \Rightarrow u^*(x)=-\phi\left(\frac{1}{2}g(x)'p\right)$$
$$d^*(x)=\frac{1}{2\gamma^2}k(x)'p. \quad (29)$$

Defining

$$H^*(x,p,u^*,d^*)=p'f-2\phi^{-1}(u^*)'u^*+h'h+2\int_0^{u^*}\phi^{-1}(v)dv+\frac{1}{4\gamma^2}p'kk'p \quad (30)$$

and rewriting (28) in terms of (30) gives

$$H(x,p,u,d)=H^*(x,p,u^*,d^*)-\gamma^2\|d-d^*\|+2\left\{\int_{u^*}^{u}\phi^{-1}(v)dv-\phi^{-1}(u^*)'(u-u^*)\right\}^2.$$

which is valid expression for all d and all $u\in U$. From Lemma 3, one has

$$H(x_0,u^*,d)\le H(x_0,u^*,d^*)\le H(x_0,u,d^*) \quad (31)$$

and Isaacs's condition follows. ■

Under regularity assumptions, from [7, Th. 2.6], there exists $V^*(x_0)\in C^1$ solving the HJI, then $V(x_0,u^*,d)\le V(x_0,u^*,d^*)\le V(x_0,u,d^*)$ and the zero-sum game has a value and the pair of policies (29) are in saddle point equilibrium.

For the infinite horizon game, as $T\to\infty$ in (27), one obtains the following Isaacs equation:

$$\begin{aligned}H^*(x,V_x,u^*,d^*)&=V_x'(f+gu^*+kd^*)+h'h\\&\quad+2\int_0^{u^*}\phi^{-1}(v)dv-\gamma^2\|d^*\|^2\\&=0\quad V(0)=0.\end{aligned} \quad (32)$$

On substitution of (29) in (32), the HJI equation for constrained input systems is obtained

$$V_x'f-V_x'g\phi\left(\frac{1}{2}g'V_x\right)+h'h+2\int_0^{-\phi\left(\frac{1}{2}g'V_x\right)}\phi^{-1}(v)dv+\frac{1}{4\gamma^2}V_x'kk'V_x=0\quad V(0)=0. \quad (33)$$

Next, it is shown that (29) remains in saddle point equilibrium as $T\to\infty$ if they are sought among finite energy strategies. See [6] and [12] for unconstrained policies.

Theorem 2: Suppose that there exists a $V(x)\in C^1$ satisfying the HJI (33) and that

$$\dot{x}=f-g\phi\left(\frac{1}{2}g'V_x\right)+\frac{1}{2\gamma^2}kk'V_x \quad (34)$$

is locally asymptotically stable, then

$$u^*(x)=-\phi\left(\frac{1}{2}g'V_x\right)\quad d^*(x)=\frac{1}{2\gamma^2}k'V_x \quad (35)$$

are in saddle point equilibrium for the infinite horizon game among strategies $u\in U$, $d\in L_2[0,\infty)$.

Proof: The proof is made by completing the squares

$$\begin{aligned}J_T(u,d;x_0)&=\int_0^T\left(h'h+\|u(t)\|_q^2-\gamma^2\|d\|^2\right)dt\\&=\int_0^T\left(h'h+\|u(t)\|_q^2-\gamma^2\|d\|^2\right)dt\\&\quad+V^*(x_0)-V^*(x_T)+\int_0^T\dot{V}^*dt\\&=\int_0^T\left(h'h+\|u(t)\|_q^2-\gamma^2\|d\|^2\right)dt\\&\quad+V^*(x_0)-V^*(x_T)+\int_0^T V_x^{*\prime}(f+gu+kd)dt\\&=\int_0^T\left(2\int_{u^*}^{u}\phi^{-1}(v)dv-2\phi^{-1}(u^*)'(u-u^*)\right.\\&\qquad\left.-\gamma^2\|d-d^*\|^2\right)dt+V^*(x_0)-V^*(x_T)\end{aligned} \quad (36)$$

where V^* solves (33). Since $u(t)$, $d(t)\in L_2[0,\infty)$, and since the game has a finite value as $T\to\infty$, this implies that $x(t)\in L_2[0,\infty)$, therefore $x(t)\to 0$, $V^*(x(\infty))=0$, and

$$\begin{aligned}J_\infty(u,d;x_0)=V^*(x_0)+\int_0^\infty&\left(2\int_{u^*}^{u}\phi^{-1}(v)dv\right.\\&\left.-2\phi^{-1}(u^*)'(u-u^*)-\gamma^2\|d-d^*\|^2\right)dt.\end{aligned} \quad (37)$$

Using Lemma 3, u^* and d^* are in saddle point equilibrium in the class of finite energy strategies. ■

Since (35) satisfies the Isaacs equation, it can be shown that the feedback saddle point is unique in the sense that it is strongly time consistent and noise insensitive [6].

Example 2: Consider the following nonlinear system

$$\dot{x}=-x^3+u+d,\qquad -1\le u\le 1$$
$$\|z\|^2=-\ln\left[1-\tanh^2(2x^3)\right]+2\int_0^u\tanh^{-1}(v)dv \quad (38)$$

Note that $h'(x)h(x) = -\ln[1-\tanh^2(2x^3)] > 0$ and is monotonically increasing in x. It follows that the HJI (33) in this case is given by

$$\begin{aligned}0 &= V_x(-x^3) + V_x\tanh(-0.5V_x) \\ &\quad + 2\int_0^{\tanh(-0.5V_x)} \tanh^{-1}(v)dv + \frac{1}{4\gamma^2}V_x^2 \\ &\quad - \ln\left[1-\tanh^2(2x^3)\right] \\ 0 &= V_x(-x^3) + V_x\tanh(-0.5V_x) \\ &\quad + 2\tanh(-0.5V_x)\tanh^{-1}(\tanh(-0.5V_x)) \\ &\quad + \ln\left[1-\tanh^2(-0.5V_x)\right] + \frac{1}{4\gamma^2}V_x^2 \\ &\quad - \ln\left[1-\tanh^2(2x^3)\right] \\ 0 &= V_x(-x^3) + \ln\left[1-\tanh^2(-0.5V_x)\right] + \frac{1}{4\gamma^2}V_x^2 \\ &\quad - \ln\left[1-\tanh^2(2x^3)\right]. \end{aligned} \tag{39}$$

Assume that $\gamma = 1$, then the available storage of the HJI equation exists and is given by $V(x) = x^4$ and the closed-loop dynamics

$$\begin{aligned}\dot{x} &= f - g\phi\left(\frac{1}{2}g'V_x\right) + \frac{1}{2\gamma^2}kk'V_x \\ &= x^3 - \tanh(2x^3)\end{aligned} \tag{40}$$

is locally asymptotically stable and, hence, the L_2- gain < 1.

Note that for arbitrary $f(x)$, $g(x)$, $k(x)$ and $h(x)$. Obtaining the analytical solution to the HJI (33) is not possible in general. In the next section, a policy iterations technique as done in Section II is proposed that reduces the solution of the HJI equation to an easier to solve iterative equation similar to (7).

V. Solving the HJI Using Policy Iterations

To solve (33) by policy iterations, we start by showing the existence and convergence of control policy iterations on the constrained input similar to work done on systems with no input constraints in [17]. Then policy iterations on both players are performed on the constrained control policy and disturbance policy.

Lemma 5: Assume that the closed-loop dynamics for the constrained stabilizing controller u_j

$$\dot{x} = f(x) + g(x)u_j + k(x)d \equiv f_j(x) + k(x)d$$

has an L_2- gain $< \gamma$ with the associated available storage $V_j \in C^1$ solving

$$V_{xj}'f_j + h'h + 2\int_0^{u_j}\phi^{-1}(v)dv + \frac{1}{4\gamma^2}V_{xj}'kk'V_{xj} = 0. \tag{41}$$

Furthermore, assume that (20) is zero-state observable. Then, the updated control policy $u_{j+1} = -\phi((1/2)g'V_{xj})$ guarantees that the closed-loop dynamics $\dot{x} = f_{j+1} + kd$ will have an $L_2 - \text{gain} \leq \gamma$ and $\dot{x} = f_{j+1}$ is asymptotically stable. It also implies that if $V_{j+1} \in C^1$, then $V_{j+1} \leq V_j$.

Proof: Note that

$$\begin{aligned}V_{xj}'f_{j+1} &= -h'h - 2\int_0^{u_{j+1}}\phi^{-1}(v)dv - \frac{1}{4\gamma^2}V_{xj}'kk'V_{xj} \\ &\quad + 2\int_{u_j}^{u_{j+1}}\phi^{-1}(v)dv - 2\phi^{-1}(u_{j+1})'(u_{j+1}-u_j).\end{aligned}$$

From Lemma 3, it follows that

$$V_{xj}'f_{j+1} + h'h + 2\int_0^{u_{j+1}}\phi^{-1}(v)dv + \frac{1}{4\gamma^2}V_{xj}'kk'V_{xj} \leq 0$$

with V_j is a possible storage for $\dot{x} = f_{j+1}$ which by zero-state observability is asymptotically stable and the available storage for $\dot{x} = f_{j+1}$ is such that $V_{j+1} \leq V_j$. ■

Theorem 3: Assume that the value function of the game is smooth $V^* \in C^1$ and solves (33) with the property that $\dot{x} = f - g\phi((1/2)g'V_x^*) + (1/2\gamma_2)kk'V_x^*$ is asymptotically stable. Assume also that $\forall j$ $\dot{x} = f_j$ is asymptotically stable with $V_j \in C^1$ solving (41) and $\dot{x} = f + gu_j + (1/2\gamma^2)kk'V_{xj}$ is asymptotically stable. Then $j \to \infty \Rightarrow \sup_{x\in\Omega_0}|V_j - V^*| \to 0$. Moreover, V^* has the largest DOV of any other constrained controller that has an $L_2 - \text{gain} < \gamma$.

Proof: From Lemma 5 $V_{j+1} \leq V_j$. Hence, V_j converges pointwise to V^* and since Ω_* is compact, uniform convergence of V_j to V^* on Ω_* follows by Dini's theorem, [3]. Since V_{j+1} is valid on Ω_j and, hence, valid on Ω_0. Therefore, V^* is valid for any Ω_0. ■

The last part of Theorem 3 implies that u^* has the largest region of asymptotic stability of any other constrained controller that is finite L_2-gain stable for a prescribed γ.

Combining Theorem 1 with Theorem 3, one obtains a two loop policy iterations solution method for the HJI (33). Specifically, select u_j, and find V_j that solves (41) by inner loop policy iterations on the disturbance as in Theorem 1 until $V_j^\infty \to V_j$ by solving

$$V_{xj}^{i\,\prime}(f_j+kd^i)+h'h+2\int_0^{u_j}\phi^{-1}(v)dv-\gamma^2\|d^i\|^2 = 0. \tag{42}$$

Then, by Theorem 3, use $u_{j+1} = -\phi((1/2)g'V_{xj}^\infty)$ in outer loop policy iterations on the constrained control.

It is important to note that one may use techniques such as neural networks to obtain a closed-form approximation of the exact solution to (42) over a domain of the state–space. See [2] for a successful implementation to the nonlinear benchmark problem.

Controllers derived using (33) for a fixed γ are *suboptimal* H_∞ controllers. *Optimal* H_∞ are achieved for the lowest possible γ^* for which the HJI is solvable. It is straightforward to show that the DOV for the game value functions $V_{\gamma_1}^*$ and $V_{\gamma_2}^*$ are such that $\Omega_{\gamma_2}^* \subseteq \Omega_{\gamma_1}^*$ for $\gamma_1 \geq \gamma_2 > \gamma^*$ with γ^* being the smallest gain for which a stabilizing solution of the HJI (33) exists.

VI. Conclusion

The constrained input HJI equation along with two players policy iterations provide a sequence of differential equations for which approximate closed-form solutions are easier to obtain. The presented method can be combined with neural networks to obtain least squares solution of the HJI equation therefore obtaining a practical method to derive L_2-gain optimal, or suboptimal H_∞, controllers of nonlinear systems that are affine in input and with actuator saturation. The method requires the problem to possess a smooth solution of the HJI equations. This is an extension to our earlier work on HJB equations [1].

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Residual Generation for Fault Diagnosis of Systems Described by Linear Differential-Algebraic Equations

Mattias Nyberg and Erik Frisk

***Abstract*—Linear residual generation for differential-algebraic equation (DAE) systems is considered within a polynomial framework where a complete characterization and parameterization of all residual generators is presented. Further, a condition for fault detectability in DAE systems is given. Based on the characterization of all residual generators, a design strategy for residual generators for DAE systems is presented. The design strategy guarantees that the resulting residual generator is sensitive to all the detectable faults and also that the residual generator is of lowest possible order. In all results derived, no assumption about observability or controllability is needed. In particular, special care has been devoted to assure the lowest-order property also for non-controllable systems.**

I. Introduction

Fault diagnosis consists of detecting and isolating faults acting on a process. In many methods, e.g., *structured residuals* [1], the concept of residuals play a central role. Commonly, a set of residuals is used where different subsets of residuals are sensitive to different subsets of faults and in this way isolation between faults is possible.

In this note, residual generation for models described by general linear differential-algebraic equations (DAEs) is considered. Previous works on residual generation have all considered more specific classes of models, i.e., transfer functions [1], [2], state-space models [3]–[5], or descriptor models e.g., [6], [7]. Since DAE models cover all these classes of models, the methods presented in this note are applicable to all the three previous cases.

In the context of residual generation, DAE-models are important because they appear in large classes of engineering systems like electrical systems, chemical processes, robotic manipulators, and mechanical systems. For example, in mechanical systems, differential equations arise from equations of motion while algebraic constraints model geometrical constraints. Further, DAE-models are also the result when using a physically based object-oriented modelling approach [8].

The approach presented in this note is an extension of the previous work [3] and one main contribution is a new method for designing residual generators for DAE-models. The method finds residual generators of lowest possible order, and which are guaranteed to be sensitive to detectable faults. Another main contribution is a criterion for fault detectability in DAE-systems, i.e., a criterion that says if it is at all possible to find any residual generator sensitive to a fault. A help in developing these results, but also a contribution on its own, is that we derive a characterization of all possible residual generators.

Previous works on residual generation for linear DAE systems have all assumed that the model is in descriptor form. As said previously, the models considered here [see (1)] are more general. However, they can with a straightforward transformation be taken to the descriptor form and, therefore, it makes sense to relate the present work to previous works dealing with descriptor models. For descriptor models, two

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The authors are with the Department of Electrical Engineering, Linköping University, SE-581 83 Linköping, Sweden (e-mail: matny@isy.liu.se; frisk@isy.liu.se).

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